Online Appendix for "Inference for Factor-MIDAS Regression Models"

Abstract

Appendix A presents the primitive assumptions necessary for proving the results in the main text. Appendices B and D provide the proofs of the results in the main text. Appendix C presents the bootstrap procedure for the factor-MIDAS regression model. Appendix E contains additional simulation results. Finally, in Appendix F, we include an additional empirical result, which is omitted from the main text.

A Primitive assumptions

This section delivers the primitive assumption for asymptotic theory. The factor-augmented MIDAS regression involves two frequencies, thus we use two time indices: $t_h = 1, \ldots, T_H$ denotes the high-frequency time index and $t = 1, \ldots, T$ denotes the low-frequency time index. Particularly, we use a subscript h to denote high-frequency time index (e.g. s_h also denotes the high-frequency time index).

Assumption A.1 (Factors and Factor Loadings)

(a) f_{t_h} *are stationary with* $E\left\|f_{t_h}\right\|^4 \leq M$ *and* $\frac{1}{T_H}\sum_{t_h=1}^{T_H} f_{t_h} f_{t_h}' \xrightarrow{p} \Sigma_f > 0$*, where* Σ_f *is a non-random* $r \times r$ *matrix.*

(b) The factor loadings λ_i *are either deterministic such that* $\|\lambda_i\| \leq M$*, or stochastic such that* $E\left\|\lambda_i\right\|^4 \leq M$*. In either case,* $\Lambda'\Lambda/N \xrightarrow{p} \Sigma_\Lambda > 0$*, where* Σ_Λ *is a non-random matrix.*

(c) The eigenvalues of the $r \times r$ *matrix* $(\Sigma_\Lambda \Sigma_f)$ *are distinct.*

(d) $f'f/T_H = I_r$ *and* $\Lambda'\Lambda$ *is a diagonal matrix with distinct entries, where* $f = (f_1, \ldots, f_{T_H})'$*.*

Assumption A.2 (Time and Cross Section Dependence and Heteroskedasticity)

(a) $E(e_{i,t_h}) = 0, E|e_{i,t_h}|^8 \leq M$*.*

(b) $E(e_{i,t_h} e_{j,s_h}) = \sigma_{ij,t_h s_h}, |\sigma_{ij,t_h s_h}| \leq \bar{\sigma}_{ij}$ *for all* (t_h, s_h) *and* $|\sigma_{ij,t_h s_h}| \leq \tau_{t_h s_h}$ *for all* (i,j) *such that* $\frac{1}{N}\sum_{i,j=1}^{N} \bar{\sigma}_{ij} \leq M$*,* $\frac{1}{T_H}\sum_{t_h,s_h=1}^{T_H} \tau_{t_h s_h} \leq M$*, and* $\frac{1}{NT_H}\sum_{t_h,s_h,i,j} |\sigma_{ij,t_h s_h}| \leq M$*.*

(c) For every (t_h, s_h)*,* $E\left|N^{-1/2}\sum_{i=1}^{N}(e_{i,t_h}e_{i,s_h} - E(e_{i,t_h}e_{i,s_h}))\right|^4 \leq M$*.*

(d) $E(e_{i,t_h}e_{j,t_h}) = \sigma_{ij}$ *and* $E(e_{i,t_h}e_{j,t_h-k}) = \sigma_{ij,k}$ *for all* t *and* k.

Assumption A.3 (Moments and Weak Dependence Among $\{f_{t_h}\}, \{\lambda_i\}$ and $\{e_{i,t_h}\}$)

(a) $E\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{\sqrt{T_H}}\sum_{t_h=1}^{T_H} f_{t_h}e_{i,t_h}\right\|^2\right) \leq M$, *where* $E(f_{t_h}e_{i,t_h}) = 0$ *for all* (i, t_h).

(b) *For each* t_h, $E\left\|\frac{1}{\sqrt{T_H N}}\sum_{s_h=1}^{T_H}\sum_{i=1}^{N} f_{s_h}(e_{i,t_h}e_{i,s_h} - E(e_{i,t_h}e_{i,s_h}))\right\|^2 \leq M$.

(c) $E\left\|\frac{1}{\sqrt{T_H N}}\sum_{t_h=1}^{T_H} f_{t_h}e'_{t_h}\Lambda\right\|^2 \leq M$, *where* $E(f_{t_h}\lambda'_i e_{i,t_h}) = 0$ *for all* (i, t_h).

(d) $E\left(\frac{1}{T_H}\sum_{t_h=1}^{T_H}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{i,t_h}\right\|^2\right) \leq M$, *where* $E(\lambda_i e_{i,t_h}) = 0$ *for all* (i, t_h).

(e) *As* $N \to \infty$, $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda'_j e_{i,t_h}e_{j,t_h} - \Gamma \xrightarrow{p} 0$ *and* $\Gamma \equiv \lim_{N\to\infty} Var\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{i,t_h}\right)$.

Assumption A.4 (Serial Dependence between $\{f_{t_h}\}$, $\{\lambda_i\}$ and $\{e_{i,t_h}\}$)

(a) $\frac{1}{T_H}\sum_{t_h=1}^{T_H} f_{t_h}f'_{t_h-k} \xrightarrow{p} \Sigma_{f,k}$, *where* $\Sigma_{f,k}$ *is a non-random* $r \times r$ *matrix.*

(b) *For each* t_h *and all* k, $E\left\|\frac{1}{\sqrt{T_H N}}\sum_{s_h=1}^{T_H}\sum_{i=1}^{N} f_{s_h}(e_{i,t_h}e_{i,s_h-k} - E(e_{i,t_h}e_{i,s_h-k}))\right\|^2 \leq M$.

(c) $E\left\|\frac{1}{\sqrt{NT_H}}\sum_{t_h=1}^{T_H} f_{t_h}e'_{t_h-k}\Lambda\right\|^2 \leq M$, *where* $E(f_{t_h}\lambda'_i e_{i,t_h-k}) = 0$ *for all* (i, t_h) *and all* k.

(d) *As* $N \to \infty$, $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda'_j e_{i,t_h}e_{j,t_h-k} - \Gamma_k \xrightarrow{p} 0$ *and* $\Gamma_k \equiv \lim_{N\to\infty} Cov\left(\frac{\Lambda' e_{t_h}}{\sqrt{N}}, \frac{\Lambda' e_{t_h-k}}{\sqrt{N}}\right)$.

Assumption A.5 (Weak Dependence Between Idiosyncratic Errors and Regression Errors)

(a) *For each* t, $E\left|\frac{1}{\sqrt{TN}}\sum_{s=1}^{T}\sum_{i=1}^{N}\varepsilon_s(e_{i,t-j/m}e_{i,s-j/m} - E(e_{i,t-j/m}e_{i,s-j/m}))\right|^2 \leq M$ *for* $j = 0, \ldots, m-1$.

(b) $E\left\|\frac{1}{\sqrt{TN}}\sum_{t=1}^{T}\sum_{i=1}^{N}\lambda_i e_{i,t-j/m}\varepsilon_t\right\|^2 \leq M$, *where* $E(\lambda_i e_{i,t-j/m}\varepsilon_t) = 0$ *for all* (i, t) *and* $j = 0, \ldots, m-1$.

Assumption A.6 (Moments and CLT for the Score Vector)

(a) $E(\varepsilon_t) = 0$ *and* $E\left|\varepsilon_t\right|^2 < M$.

(b) $E\left\|g_{\alpha,t}\right\|^4 \leq M$ *and* $\frac{1}{T}\sum_{t=1}^{T} g_{\alpha,t}g'_{\alpha,t} \xrightarrow{p} \Sigma > 0$ *where* $g_{\alpha,t} = \partial g(F_t, \alpha)/\partial\alpha$.

(c) *As* $T \to \infty$, $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_{\alpha,t}\varepsilon_t \xrightarrow{d} N(0, \Omega)$, *where* $E\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_{\alpha,t}\varepsilon_t\right\|^2 < M$ *and* $\Omega \equiv \lim_{T\to\infty} Var\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g_{\alpha,t}\varepsilon_t\right) > 0$.

Assumption A.1 are standard assumptions on the factors and the factor loadings in the factor analysis. Additionally, we assume that the factors are stationary. This is to allow $\Sigma_f = \text{plim}\, \frac{1}{T_H}\sum_{t_h=1}^{T_H} f_{t_h}f'_{t_h} = \text{plim}\, \frac{1}{T}\sum_{t=1}^{T} f_{t-j/m}f'_{t-j/m}$, for all j. Assumption A.1-(d) is one of the identifying restrictions from Bai and Ng (2013). By imposing this assumption, the rotation matrix H_0 is a diagonal matrix of ± 1, where the sign is determined by $\tilde{f}'f/T_H$. However, since the true factors are unknown, we still do not know the sign of the rotation matrix.

Assumption A.2 and Assumption A.3 can be found equivalently in Gonçalves and Perron (2014) (henceforth, GP (2014)) (their Assumptions 2 and 3, respectively). In Assumption A.2, we allow weak cross-sectional and serial dependence in the idiosyncratic error terms. In Assumption A.3, we impose some moment condition between the factors, idiosyncratic error terms, and the factor loadings. We also allow some weak dependence among them. Due to the MIDAS structure, where the lags of the factors are used, we also allow some serial dependence between them in Assumption A.4. This set of assumptions is new in the context of the factor-augmented regression models. In particular, Assumption A.4-(d) allows for the serial dependence in the scaled average over cross-sectional dimension of factor loadings and idiosyncratic error term in the factor model.

We impose some weak dependence between idiosyncratic error terms and the regression errors in Assumption A.5. This Assumption is equivalent to the Assumption 4 in GP (2014). Assumption A.6 imposes some moment condition on $\{\varepsilon_t\}$ and the score vector $g_{\alpha,t}$. Assumption A.6-(b) requires that we can apply a law of large numbers on $\{g_{\alpha,t}g'_{\alpha,t}\}$. By introducing Assumption A.6-(c), we can apply a central limit theorem on $\{g_{\alpha,t}\varepsilon_t\}$. Similar assumptions to Assumption A.5 and A.6 can be found in GP (2014).

B Proof of results in Section 2

In this section, we prove the asymptotic distribution of NLS estimators in Theorem 2.1 and Theorem 2.2, the consistency of the variance-covariance of the cross-sectional average of the factor loadings and idiosyncratic error term across time for the plug-in bias estimator. To prove the asymptotic distribution, we use the following lemmas. The proof for the following lemmas Lemma B.1 to Lemma B.3 can be found at the end of proof of Theorem 2.1.

Lemma B.1 $\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t(\tilde{F}_t(\theta) - HF_t(\theta)) = o_p(1)$.

Lemma B.2 *For $j, l = 0, \ldots, m-1$, if $\sqrt{T}/N \to c$, where $0 \le c < \infty$,*

(a) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}_{t-j/m} - HF_{t-j/m})(\tilde{f}_{t-j/m} - Hf_{t-j/m})' = cV^{-1}H\Gamma HV^{-1} + o_p(1)$,

(b) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}_{t-j/m} - Hf_{t-j/m})(\tilde{f}_{t-l/m} - Hf_{t-l/m})' = cV^{-1}H\Gamma_{j-l}HV^{-1} + o_p(1)$ *for* $j \neq l$,

(c) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}Hf_{t-j/m}(\tilde{f}_{t-j/m} - Hf_{t-j/m})' = cH\Gamma Q'V^{-2} + o_p(1)$,

(d) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}Hf_{t-l/m}(\tilde{f}_{t-j/m} - Hf_{t-j/m})' = cQ_{j-l}\Gamma Q'V^{-2} + o_p(1)$ *for* $j \neq l$.

Lemma B.3 *If $\sqrt{T}/N \to c$, where $0 \le c < \infty$,*

(a) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{F}_t(\theta)-HF_t(\theta))(\tilde{F}_t(\theta)-HF_t(\theta))'$

$= cV^{-1}Q\left\{\sum_{k=1}^{K}w_k(\theta)\Gamma w_k(\theta)+\sum_{k=1}^{K}\sum_{l\neq k}^{K}w_k(\theta)\Gamma_{k-l}w_l(\theta)\right\}Q'V^{-1}+o_p(1),$

(b) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{F}_t(\theta)-HF_t(\theta))(HF_t(\theta))'$

$= c\left\{\sum_{k=1}^{K}w_k^2(\theta)H+\sum_{k=1}^{K}\sum_{l\neq k}^{K}w_k(\theta)Q_{k-l}w_l(\theta)\right\}\Gamma Q'V^{-2}+o_p(1).$

Note that we write $F_t(\theta)=\sum_{k=1}^{K}w_k(\theta)f_{t-k/m}$, where $w_k(\theta)\equiv \text{diag}(w_{k,1}(\theta_1),\dots,w_{k,r}(\theta_r))$ is a $r\times r$ diagonal matrix. We also define $\delta_{NT_H}=\min(\sqrt{N},\sqrt{T_H})$. We first prove Theorem 2.1 and then we prove Lemmas B.1 - B.3.

Proof of Theorem 2.1. As the NLS estimators $\tilde{\alpha}$ maximizes the objective function $\tilde{Q}_T(\alpha)=-\frac{1}{T}\sum_{t=1}^{T}[y_t-g(\tilde{F}_t,\alpha)]^2$, we have

$$\sqrt{T}(\tilde{\alpha}-\alpha)=-\Big[\frac{1}{T}\sum_{t=1}^{T}H(\tilde{F}_t,\alpha_T)\Big]^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}s(\tilde{F}_t,\alpha), \quad (1)$$

where α_T is the intermediate between $\tilde{\alpha}$ and α and $H(\tilde{F}_t,\alpha)$ is a hessian matrix and $s(\tilde{F}_t,\alpha)$ is a score vector. For deriving the asymptotic distribution, we analyse the convergence of each term. Let $g_\alpha(\cdot)=\partial g(\cdot)/\partial\alpha$. We write the term with a score vector as follows.

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}s(\tilde{F}_t,\alpha)&=2\frac{1}{\sqrt{T}}\sum_{t=1}^{T}[\varepsilon_t+\beta'H^{-1}(HF_t(\theta)-\tilde{F}_t(\theta))]g_\alpha(\tilde{F}_t,\alpha)\\&=2\frac{1}{\sqrt{T}}\sum_{t=1}^{T}[\varepsilon_t+\beta'H^{-1}(HF_t(\theta)-\tilde{F}_t(\theta))](\Phi_0 g_\alpha(F_t,\alpha)+P_t),\end{aligned}$$

where where $\Phi_0=\text{diag}(1,H_0,I_p)$ and $H_0=\text{plim}\,H$ and P_t is a $(1+r+p)\times 1$ vector such that

$$P_t=\begin{bmatrix}0\\ \tilde{F}_t(\theta)-HF_t(\theta)\\ \left(\frac{\partial\tilde{F}_t(\theta)}{\partial\theta}H^{-1}-\frac{\partial F_t(\theta)}{\partial\theta}\right)'\beta\end{bmatrix},$$

with $\frac{\partial \tilde{F}_t(\theta)'}{\partial \theta} = \text{diag}\left(\frac{\partial \tilde{F}_{1,t}(\theta_1)}{\partial \theta_1}, \ldots, \frac{\partial \tilde{F}_{r,t}(\theta_r)}{\partial \theta_r}\right)$ is a $r \times r$ block-diagonal matrix. k-th block is $\partial \tilde{F}_{k,t}(\theta_k)/\partial \theta_k$, which is a $p_j \times 1$ column vector, for $j = 1, \ldots, r$. Under Assumption A.6 and Lemma B.1, we have $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t g_\alpha(\tilde{F}_t, \alpha) \xrightarrow{d} N(0, \Phi_0 \Omega \Phi_0')$. The remaining term drives the bias in Theorem 2.1. Note that the bias exists in the slope coefficients β_1 and the weighting parameters θ. With respect to β_1, the remaining term is as follows.

$$
\begin{aligned}
&\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{F}_t(\theta)[HF_t(\theta) - \tilde{F}_t(\theta)]' H^{-1'} \beta_1 \\
&= -\left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\tilde{F}_t(\theta) - HF_t(\theta))(\tilde{F}_t(\theta) - HF_t(\theta))' + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} HF_t(\theta)(\tilde{F}_t(\theta) - HF_t(\theta))'\right] H^{-1'} \beta_1 \\
&= -c \left[V^{-1} H \left\{\sum_{k=1}^{K} w_k(\theta) \Gamma w_k(\theta) + \sum_{k=1}^{K} \sum_{l \neq k}^{K} w_k(\theta) \Gamma_{k-l} w_l(\theta)\right\} HV^{-1} \right. \\
&\qquad \left. + \left\{\sum_{k=1}^{K} w_k(\theta) H w_k(\theta) + \sum_{k=1}^{K} \sum_{l \neq k}^{K} w_k(\theta) Q_{k-l} w_l(\theta)\right\} \Gamma Q' V^{-2}\right] \text{plim}(\tilde{\beta}_1) \\
&= -c B_{\beta_1} + o_p(1), \qquad (2)
\end{aligned}
$$

where $\text{plim}(\tilde{\beta}_1) = H^{-1'} \beta_1$. The second equality follows by applying Lemma B.3. Similarly, with respect to θ, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial \tilde{F}_t(\theta)'}{\partial \theta} H^{-1'} \beta_1 \beta_1' H^{-1} [HF_t(\theta) - \tilde{F}_t(\theta)] \\
&= -H^{-1'} \beta_1 \circ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{F}_{t,\theta}(\theta) [\tilde{F}_t(\theta) - HF_t(\theta)]' H^{-1'} \beta_1 \\
&= -c \, \text{plim}(\tilde{\beta}_1) \circ \left[V^{-1} H \left\{\sum_{k=1}^{K} \frac{\partial w_k(\theta)}{\partial \theta} \Gamma w_k(\theta) + \sum_{k=1}^{K} \sum_{l \neq k}^{K} \frac{\partial w_k(\theta)}{\partial \theta} \Gamma_{k-l} w_l(\theta)\right\} HV^{-1} \right. \\
&\qquad \left. + \left\{\sum_{k=1}^{K} \frac{\partial w_k(\theta)}{\partial \theta} H w_k(\theta) + \sum_{k=1}^{K} \sum_{l \neq k}^{K} \frac{\partial w_k(\theta)}{\partial \theta} Q_{k-l} w_l(\theta)\right\} \Gamma Q' V^{-2}\right] \text{plim}(\tilde{\beta}_1) \\
&= -c B_\theta + o_p(1), \qquad (3)
\end{aligned}
$$

where $\tilde{F}_{t,\theta}(\theta) = \left(\frac{\partial \tilde{F}_{1,t}(\theta_1)}{\partial \theta_1}, \ldots, \frac{\partial \tilde{F}_{r,t}(\theta_r)}{\theta_r}\right)'$. To apply the lemmas, we use the Hadamard product such that $(A \circ B)_{ij} = A_{ij}B_{ij}$. By applying Hadamard product, we have $\frac{\partial \tilde{F}_t(\theta)'}{\partial \theta} H^{-1'}\beta = H^{-1'}\beta \circ \tilde{F}_{t,\theta}(\theta)$ to obtain the first equality. Then, we apply Lemma B.3 for the second equality. Finally, we have $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} s(\tilde{F}_t, \alpha) \xrightarrow{d} N(-cB_\alpha, \Phi_0 \Omega \Phi_0')$. Next, we derive the term with Hessian matrix. First, we rewrite the first term in (1) as follows.

$$\frac{1}{T}\sum_{t=1}^{T} H(\tilde{F}_t, \alpha) = \frac{1}{T}\sum_{t=1}^{T}\left[\varepsilon_t + \beta' H^{-1}(HF_t(\theta) - \tilde{F}_t(\theta))\right]\frac{\partial^2 g(\tilde{F}_t, \alpha)}{\partial \alpha \partial \alpha'} + \frac{1}{T}\sum_{t=1}^{T}\frac{\partial g(\tilde{F}_t, \alpha)}{\partial \alpha}\frac{\partial g(\tilde{F}_t, \alpha)}{\partial \alpha'}.$$

Under Assumption A.6 and Lemma B.1, $\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t \frac{\partial^2 g(\tilde{F}_t,\alpha)}{\partial \alpha \partial \alpha'} = o_p(1)$. We can also show that $-\frac{1}{T}\sum_{t=1}^{T}\beta' H^{-1}(\tilde{F}_t(\theta) - HF_t(\theta))\frac{\partial^2 g(\tilde{F}_t,\alpha)}{\partial \alpha \partial \alpha'} = o_p(1)$. Finally, for the second term, we have

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial g(\tilde{F}_t, \alpha)}{\partial \alpha}\frac{\partial g(\tilde{F}_t, \alpha)}{\partial \alpha'} = \Phi_0 \Sigma \Phi_0' + o_p(1) \tag{4}$$

where $\Sigma \equiv E\left[\frac{\partial g(F_t,\alpha)}{\partial \alpha}\frac{\partial g(F_t,\alpha)}{\partial \alpha'}\right]$ by replacing $\frac{\partial g(\tilde{F}_t,\alpha)}{\partial \alpha}$ with $\Phi_0 \frac{\partial g(F_t,\alpha)}{\partial \alpha} + P_t$. Then, by Lemma B.2, we have $\frac{1}{T}\sum_{t=1}^{T} g_\alpha(F_t, \alpha)P_t' = o_p(1)$ and $\frac{1}{T}\sum_{t=1}^{T} P_t P_t' = o_p(1)$. By plugging the terms, (2), (3), and (4) into (1), we have $\sqrt{T}(\tilde{\alpha} - \alpha) \xrightarrow{d} N(-c(\Phi_0 \Sigma \Phi_0')^{-1}B_\alpha, \Phi_0'^{-1}\Sigma^{-1}\Omega\Sigma^{-1}\Phi_0^{-1})$. ■

Next, we prove Lemma B.1-B.3, which we used to prove Theorem 2.1. We can obtain Lemma B.1 by applying the arguments in the proof of Lemma 1.1 in GP (2014). The proofs for (a) and (c) in Lemma B.2 are also similar to the proof of Lemma A.2 - (a) and (b) in GP (2014). Therefore, here, we show the proof for (b) and (d) in Lemma B.2. While we employ similar arguments to those in GP (2014) to prove (b) and (d), our proof relies on a new set of assumption, specifically Assumption A.4. This highlights the importance to account for serial dependence in the idiosyncratic error term in our framework, representing a novel contribution to the literature.

Proof of Lemma B.2 - (b). First, we use the identity for the factor estimation error in GP (2014) such that $\tilde{f}_{t_h} - Hf_{t_h} = \tilde{V}^{-1}(A_{1,t_h} + A_{2,t_h} + A_{3,t_h} + A_{4,t_h})$, where $A_{1,t_h} = \frac{1}{T_H}\sum_{s_h=1}^{T_H} \tilde{f}_{s_h}\gamma_{s_h t_h}$, $A_{2,t_h} = \frac{1}{T_H}\sum_{s_h}^{T_H} \tilde{f}_{s_h}\zeta_{s_h t_h}$, $A_{3,t_h} = \frac{1}{T_H}\sum_{s_h}^{T_H} \tilde{f}_{s_h}\eta_{s_h t_h}$, and $A_{4,t_h} = \frac{1}{T_H}\sum_{s_h}^{T_H} \tilde{f}_{s_h}\xi_{s_h t_h}$. Each term in A_{i,t_h} for $i = 1,2,3,4$ denotes the following: $\gamma_{s_h t_h} = E\left(\frac{1}{N}\sum_{i=1}^{N} e_{i,s_h} e_{i,t_h}\right)$, $\zeta_{s_h t_h} = \frac{1}{N}\sum_{i=1}^{N}(e_{i,s_h}e_{i,t_h} - E(e_{i,s_h}e_{i,t_h}))$, $\eta_{s_h t_h} = f'_{s_h}\frac{\Lambda' e_{t_h}}{N}$, and $\xi_{s_h t_h} = f'_{t_h}\frac{\Lambda' e_{s_h}}{N} = \eta_{t_h s_h}$. Under this identity and using the low-frequency notation, we have

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}_{t-j/m} - Hf_{t-j/m})(\tilde{f}_{t-l/m} - Hf_{t-l/m})' \\
&\quad = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Big[\tilde{V}^{-1}(A_{1,t-j/m} + A_{2,t-j/m} + A_{3,t-j/m} + A_{4,t-j/m}) \\
&\qquad \times (A_{1,t-l/m} + A_{2,t-l/m} + A_{3,t-l/m} + A_{4,t-l/m})'\tilde{V}^{-1}\Big],
\end{aligned}$$

for $j = 1, \ldots, m-1$. We analyse the convergence limit of each term, respectively. The proof is similar to the proof of Lemma A.2 - (a) in GP (2014). By applying the Cauchy-Schwarz inequality, we have $\left\|\frac{1}{T}\sum_{t=1}^{T} A_{1,t-j/m}A'_{1,t-l/m}\right\| \leq \left(\frac{1}{T}\sum_{t=1}^{T}\|A_{1,t-j/m}\|^2\right)^{1/2}\left(\frac{1}{T}\sum_{t=1}^{T}\|A_{1,t-l/m}\|^2\right)^{1/2} = O_p(1/T)$, by Assumptions A.1 and A.2. This implies $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} A_{1,t-j/m}A'_{1,t-l/m} = o_p(1)$. We can also show that $\left\|\frac{1}{T}\sum_{t=1}^{T} A_{2,t-j/m}A'_{2,t-l/m}\right\| \leq \left(\frac{1}{T}\sum_{t=1}^{T}\left\|A_{2,t-j/m}\right\|^2\right)^{1/2}\left(\frac{1}{T}\sum_{t=1}^{T}\left\|A_{2,t-l/m}\right\|^2\right)^{1/2} = O_p(N^{-1}\delta_{NT_H}^{-2})$ by Cauchy-Schwarz. We also use $\frac{1}{T}\sum_{t=1}^{T}\|A_{2,t-j/m}\|^2 = O_p(N^{-1}\delta_{NT_H}^{-2})$ by Assumption A.2 and $\frac{1}{T_H}\sum_{s_h=1}^{T_H}\|\tilde{f}_s - Hf_s\|^2 = O_p(\delta_{NT_H}^{-2})$ in Bai and Ng (2006). Again, this implies $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} A_{2,t-j/m}A'_{2,t-l/m} = o_p(1)$. Similarly, we can show all the terms are negligible, except the term $\frac{1}{T}\sum_{t=1}^{T} A_{3,t-j/m}A'_{3,t-l/m}$. In fact, this term is $O_p(1/N)$, which is non-negligible when it is multiplied by $\sqrt{T}$ under our assumption, $\sqrt{T}/N \to c$. To see this, we first rewrite the term as follows.

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} A_{3,t-j/m}A'_{3,t-l/m} &= \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{T_H}\sum_{t=1}^{T_H}(\tilde{f}_s - Hf_s + Hf_s)\eta_{s,t-j/m}\right)\left(\frac{1}{T_H}\sum_{s=1}^{T_H}(\tilde{f}_s - Hf_s + Hf_s)\eta_{s,t-l/m}\right)' \\
&= b_{33.1} + b_{33.2} + b'_{33.2} + b_{33.3}
\end{aligned}$$

The first term $b_{33.1}$ is bounded by $\left(\frac{1}{T_H}\sum_{s=1}^{T_H}\|\tilde{f}_s - Hf_s\|^2\right)\left(\frac{1}{TT_H}\sum_{t=1}^{T}\sum_{s=1}^{T_H}|\eta_{s,t-j/m}\eta_{s,t-l/m}|\right)$ by applying Cauchy-Schwarz inequality. This is $O_p(N^{-1}\delta_{NT_H}^{-2})$ by $\frac{1}{TT_H}\sum_{t=1}^{T}\sum_{s_h=1}^{T_H}|\eta_{s_h,t-j/m}|^2 = O_p(N^{-1})$ under Assumption A.3. Similarly, the second term is bounded by Cauchy-Schwarz such that $b_{33.2} \leq \left(\frac{1}{T_H}\sum_{s=1}^{T_H}\left\|Hf_s(\tilde{f}_s - Hf_s)\right\|\right)\left(\frac{1}{TT_H}\sum_{t=1}^{T}\sum_{s=1}^{T_H}|\eta_{s,t-j/m}\eta_{s,t-l/m}|\right) = O_p(N^{-1}\delta_{NT_H}^{-1})$. Then, the final term is $b_{33.3} = H\left(\frac{f'f}{T_H}\right)\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\Lambda' e_{t-j/m}}{N}\right)\left(\frac{e'_{t-l/m}\Lambda}{N}\right)\right]\left(\frac{f'f}{T_H}\right)H' = O_p(N^{-1})$ by Assumption A.3. Thus,

$$\sqrt{T}b_{33.3} = \frac{\sqrt{T}}{N}H\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\Lambda' e_{t-j/m}}{\sqrt{N}}\right)\left(\frac{e'_{t-l/m}\Lambda}{\sqrt{N}}\right)\right]H = cH\Gamma_{j-l}H + o_p(1),$$

where we use $\frac{f'f}{T_H} = I_r$ by Assumptions A.1-(d) and A.4-(d). Finally, we have $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}_{t-j/m} - Hf_{t-j/m})(\tilde{f}_{t-l/m} - Hf_{t-l/m}^{(m)})' = cV^{-1}H\Gamma_{j-l}HV^{-1} + o_p(1)$. ■

Proof of Lemma B.2 - (d). The proof is similar to the proof of Lemma A.2 - (b) in GP (2014). By using the identity we use in the proof of B.2-(b), we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}Hf_{t-l/m}(\tilde{f}_{t-j/m} - Hf_{t-j/m})' = H\frac{1}{\sqrt{T}}\sum_{t=1}^{T}f_{t-l/m}(A_{1,t-j/m} + A_{2,t-j/m} + A_{3,t-j/m} + A_{4,t-j/m})'\tilde{V}^{-1}$$
$$\equiv \sqrt{T}H(d_{f1} + d_{f2} + d_{f3} + d_{f4})\tilde{V}^{-1}.$$

We show the convergence limit for d_{fi}, for $i = 1, 2, 3, 4$. We can show that all the terms except d_{f4} is negligible. For example, $d_{f1} = O_p(\delta_{NT_H}^{-1}T^{-1/2}) + O_p(T_H^{-1})$. To show this, we first rewrite d_{f1} as $\frac{1}{T}\sum_{t=1}^{T}f_{t-l/m}\left(\frac{1}{T_H}\sum_{s=1}^{T_H}(\tilde{f}_s - Hf_s)'\gamma_{s,t-j/m}\right) + \frac{1}{T}\sum_{t=1}^{T}f_{t-l/m}\left(\frac{1}{T_H}\sum_{s=1}^{T_H}f'_s\gamma_{s,t-j/m}\right)H'$. The first term of d_{f1} is $O_p(\delta_{NT_H}^{-1}T^{-1/2})$ by applying Assumptions A.1-A.2 and $\frac{1}{T_H}\sum_{s_h=1}^{T_H}\|\tilde{f}_s - Hf_s\|^2 = O_p(\delta_{NT_H}^{-2})$. The second term is $O_p(T_H^{-1})$ by Cauchy-Schwarz inequality and Assumptions A.1 and A.2. We can also show that $\|d_{f2}\| = O_p((TN)^{-1/2})$ by showing $\frac{1}{T_H}\sum_{s=1}^{T_H}\left\|\frac{1}{T}\sum_{t=1}^{T}f_{t-l/m}\zeta_{s,t-j/m}\right\|^2 = O_p((TN)^{-1})$ under Assumption A.4-(b). The third term is also bounded by Cauchy-Schwarz inequality such that $\|d_{f3}\| = O_p((NT)^{-1/2})$ and by applying Assumption A.4-(c). Finally, we decompose the last term into two parts as follows.

$$d_{f4} = \frac{1}{T}\sum_{t=1}^{T} f_{t-l/m}\left(\frac{1}{T_H}\sum_{s=1}^{T_H}(\tilde{f}_s - Hf_s)'\xi_{s,t-j/m}\right) + \frac{1}{T}\sum_{t=1}^{T} f_{t-l/m}\left(\frac{1}{T_H}\sum_{s=1}^{T_H} f_s'\xi_{s,t-j/m}\right)H'$$
$$\equiv d_{f4.1} + d_{f4.2}.$$

By rearranging the second term, we have $d_{f4.2} = \frac{1}{\sqrt{T_H N}}\left(\frac{1}{T}\sum_{s=1}^{T} f_{t-l/m}f'_{t-j/m}\right)\left(\frac{1}{\sqrt{T_H N}}\sum_{s=1}^{T_H}\Lambda' e_s f_s'\right) = O_p(1/(\sqrt{T_H N}))$ by Assumptions A.4-(1) and A.3-(c). We can also rearrange the terms in the first term and write it as follows.

$$\begin{aligned} d_{f4.1} &= \frac{1}{T}\sum_{t=1}^{T} f_{t-l/m}\left[\frac{1}{T_H}\sum_{s=1}^{T_H}(\tilde{f}_s - Hf_s)'\left(f'_{t-j/m}\frac{\Lambda' e_s}{N}\right)\right] \\ &= \left(\frac{1}{T}\sum_{t=1}^{T} f_{t-l/m}f'_{t-j/m}\right)\left(\frac{1}{T_H}\sum_{s=1}^{T_H}\frac{\Lambda' e_s}{N}(\tilde{f}_s - Hf_s)'\right). \end{aligned}$$

This is $O_p(1/N)$ under our assumptions. By using $\frac{1}{T_H}\sum_{s=1}^{T_H}\frac{\Lambda' e_s}{N}(\tilde{f}_s - Hf_s)' = \frac{1}{N}(\Gamma + o_p(1))Q'V^{-1}$, from the proof in GP (2014), we have

$$\sqrt{T}Hd_{f4.1} = H\left(\frac{1}{T}\sum_{t=1}^{T} f_{t-l/m}f'_{t-j/m}\right)\left(\frac{\sqrt{T}}{N}(\Gamma + o_p(1))Q'V^{-1}\right) = cQ_{j-l}\Gamma Q'V^{-1} + o_p(1)$$

Thus, $\sqrt{T}d_{f4.1}\tilde{V}^{-1} = cQ_{j-l}\Gamma Q'V^{-2} + o_p(1)$, where $Q_{j-l} = \frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-j/m}f_{t-l/m} = \frac{1}{T_H}\sum_{t=1}^{T_H}\tilde{f}_t f_{t-(j-l)}$.

■

Proof of Lemma B.3 - (a). We write the equation as follows.

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{F}_t(\theta)-F_t(\theta))(\tilde{F}_t(\theta)-HF_t(\theta))' \\
&=\frac{1}{\sqrt{T}}\left[\sum_{j=1}^{K}w_j(\theta)(\tilde{f}_{t-j/m}-Hf_{t-j/m})\right]\left[\sum_{j=1}^{K}w_j(\theta)(\tilde{f}_{t-j/m}-Hf_{t-j/m})\right]' \\
&=\sum_{j=1}^{K}w_j(\theta)\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}_{t-j/m}-Hf_{t-j/m})(\tilde{f}_{t-j/m}-Hf_{t-j/m})'\right]w_j(\theta) \\
&\quad+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\theta)\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}_{t-j/m}-Hf_{t-j/m})(\tilde{f}_{t-l/m}-Hf_{t-l/m})'\right]w_l(\theta) \\
&=cV^{-1}Q\left\{\sum_{j=1}^{K}w_j^2(\theta)\Gamma+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\theta)\Gamma_{j-l}w_l(\theta)\right\}Q'V^{-1}+o_p(1).
\end{aligned}$$

By applying Lemmas B.2-(a) and (b), the result follows immediately. ■

Proof of Lemma B.3 - (b). Similar to previous proof, we rewrite the equation as follows.

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}HF_t(\theta)(\tilde{F}_t(\theta)-HF_t(\theta))' \\
&=\frac{1}{\sqrt{T}}\left[\sum_{j=1}^{K}w_j(\theta)Hf_{t-j/m})\right]\left[\sum_{j=1}^{K}w_j(\theta)(\tilde{f}_{t-j/m}-Hf_{t-j/m})\right]' \\
&=\sum_{j=1}^{K}w_j(\theta)\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}Hf_{t-j/m}(\tilde{f}_{t-j/m}-Hf_{t-j/m})'\right]w_j(\theta) \\
&\quad+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\theta)\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}Hf_{t-l/m}(\tilde{f}_{t-j/m}-Hf_{t-j/m})'\right]w_l(\theta) \\
&=c\left\{\sum_{j=1}^{K}w_j^2(\theta)H+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\theta)Q_{j-l}w_l(\theta)\right\}\Gamma Q'V^{-2}+o_p(1).
\end{aligned}$$

By applying Lemmas B.2-(c) and (d), the result follows. ■

Next, we prove Theorem 2.2 and Proposition 2.1. To prove Theorem 2.2, we first prove the case when there is no cross-sectional dependence (only serial correlation) in the idiosyncratic term in the factor model, and then we prove when the cross-sectional dependence is allowed.

Proof of Theorem 2.2.

If the idiosyncratic terms are serially correlated, but not cross-sectionally correlated, note that $\Gamma_k = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \lambda_i \lambda_i' E(e_{i,t_h} e_{i,t_h-k})$. Recall that the estimator for Γ_k under serial dependence without cross-sectional dependence is $\hat{\Gamma}_k = \frac{1}{N(T_H-k)}\sum_{t_h=k+1}^{T_H}\sum_{i=1}^{N} \tilde{\lambda}_i \tilde{\lambda}_i' \tilde{e}_{i,t_h} \tilde{e}_{i,t_h-k}$. To show that $\hat{\Gamma}_k - H_0^{-1\prime}\Gamma_k H_0^{-1} \to 0$, we can use the arguments in the proof of Theorem 6 in Bai (2003). In fact, we can use the fact that $\tilde{e}_{i,t_h} = e_{i,t_h} + O_p(\delta_{NT_H}^{-1})$ and $\tilde{\lambda}_i = H^{-1\prime}\lambda_i + O_p(\delta_{NT_H}^{-1})$, and rewrite $\hat{\Gamma}_k$ as follows.

$$\hat{\Gamma}_k = H^{-1\prime} \frac{1}{N(T_H-k)} \sum_{t_h=k+1}^{T_H} \sum_{i=1}^{N} \lambda_i \lambda_i' e_{i,t_h} e_{i,t_h-k} H^{-1} + o_p(1).$$

Since we have $\frac{1}{T_H-k}\sum_{t_h=k+1}^{T_H} e_{i,t_h} e_{i,t_h-k} \to E(e_{i,t_h} e_{i,t_h-k})$ and $H \to H_0$, we can show that $\hat{\Gamma}_k - H_0^{-1\prime}\Gamma_k H_0^{-1} \xrightarrow{p} 0$.

Next, we prove the case when the idiosyncratic terms are serially and cross-setionally correlated, we can use the arguments in the proof of Theorem 4 in Bai and Ng (2006). Under Assumption A.2 - (d), we have $\sigma_{ij,k} = E(e_{i,t_h} e_{j,t_h-k})$. Let $\tilde{\sigma}_{ij,k} = \frac{1}{T_H-k}\sum_{t_h=k+1}^{T_H} \tilde{e}_{i,t_h} \tilde{e}_{i,t_h-k}$ and $\Gamma_{n,k} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij,k} \lambda_i \lambda_j'$. By the definition, $\Gamma_k = \lim_{n\to\infty} \Gamma_{n,k}$. Let $\bar{\Gamma}_{n,k} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{\sigma}_{ij,k} \lambda_i \lambda_j'$. Then, we can write

$$\hat{\Gamma}_k - H^{-1\prime}\Gamma_k H^{-1} = \hat{\Gamma}_k - H^{-1\prime}\bar{\Gamma}_{n,k} H^{-1} + H^{-1\prime}(\bar{\Gamma}_{n,k} - \Gamma_{n,k}) H^{-1} + H^{-1\prime}(\Gamma_{n,k} - \Gamma_k) H^{-1}.$$

Since Γ_k is the limit of $\Gamma_{n,k}$, we have $\Gamma_{n,k} - \Gamma_k \to 0$. The remaining parts to show are $\bar{\Gamma}_{n,k} - \Gamma_{n,k} \xrightarrow{p} 0$ if $n/N \to 0$ and $n/T_H \to 0$ and $\hat{\Gamma}_k - H^{-1\prime}\bar{\Gamma}_{n,k} H^{-1} \xrightarrow{p} 0$. We first rewrite

$\bar{\Gamma}_{n,k} - \Gamma_{n,k}$ as follows.

$$
\begin{aligned}
\bar{\Gamma}_{n,k} - \Gamma_{n,k} &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(\tilde{\sigma}_{ij,k} - \sigma_{ij,k})\lambda_i\lambda_j' \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H}(e_{i,t_h}e_{j,t_h-k} - \sigma_{ij,k})\lambda_i\lambda_j' \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H} e_{i,t_h}(c_{j,t_h-k} - \tilde{c}_{j,t_h-k})\lambda_i\lambda_j' \\
&\quad - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H} e_{j,t_h-k}(c_{i,t_h} - \tilde{c}_{i,t_h})\lambda_i\lambda_j' \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H} (c_{i,t_h} - \tilde{c}_{i,t_h})(c_{j,t_h-k} - \tilde{c}_{j,t_h-k})\lambda_i\lambda_j' \\
&= I + II + III + IV,
\end{aligned}
$$

where we obtain the second equality by using the decomposition such that $\tilde{e}_{i,t_h}\tilde{e}_{j,t_h-k} = e_{i,t_h}e_{j,t_h-k} - e_{i,t_h}(c_{j,t_h-k} - \tilde{c}_{j,t_h-k}) - e_{j,t_h-k}(c_{i,t_h} - \tilde{c}_{i,t_h}) + (c_{i,t_h} - \tilde{c}_{i,t_h})(c_{j,t_h-k} - \tilde{c}_{j,t_h-k})$, where $\tilde{c}_{i,t_h} = \tilde{\lambda}_i'\tilde{f}_{t_h}$ and $c_{i,t_h} = \lambda_i' f_{t_h}$. We can show that I is $O_p((T_H - k)^{-1/2})$ since it is zero mean process. By using $c_{j,t_h} - \tilde{c}_{j,t_h} = (H^{-1'}\lambda_j - \tilde{\lambda}_j)'\tilde{f}_{t_h} + \lambda_j' H^{-1}(Hf_{t_h} - \tilde{f}_{t_h})$ and following Bai and Ng (2006), we have $II \to 0$ if $\sqrt{n}/T_H \to 0$ and $n/\delta^2_{NT_H} \to 0$. Similarly, we have $III \to 0$ as $n/\delta^2_{NT_H} \to 0$. Finally, for IV, by Cauchy-Schwarz inequality, we have

$$
\|IV\| \le \left(\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(c_{i,t_h} - \tilde{c}_{i,t_h})\lambda_i\right\|^2\right)^{1/2}\left(\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(c_{j,t_h-k} - \tilde{c}_{j,t_h-k})\lambda_j\right\|^2\right)^{1/2}
$$

Since $c_{i,t_h} - \tilde{c}_{i,t_h} = (H^{-1}\lambda_i - \tilde{\lambda}_i)'\tilde{f}_{t_h} + \lambda_i' H^{-1}(Hf_{t_h} - \tilde{f}_{t_h})$, by using c_r inequality,

$$
\begin{aligned}
\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(c_{i,t_h} - \tilde{c}_{i,t_h})\lambda\right\|^2 &\le 2\left(\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H}\|f_{t_h}\|^2\right)\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_i(H^{-1'}\lambda_i - \tilde{\lambda}_i)'\right\|^2 \\
&\quad + 2\left\|H^{-1}\right\|^2\left(\frac{1}{n}\sum_{i=1}^{n}\|\lambda_i\|^2\right)^2\frac{1}{T_H - k}\sum_{t_h=k+1}^{T_H}\left\|\tilde{f}_{t_h} - Hf_{t_h}\right\|^2.
\end{aligned}
$$

The first term term and the second term converge to zero as $\sqrt{n}/T \to 0$ and $n/T_H \to 0$. The last remaining term is $\hat{\Gamma}_k - H^{-1'}\bar{\Gamma}_{n,k}H^{-1}$. We can rewrite this term as follows.

$$
\begin{aligned}
\hat{\Gamma}_k - H^{-1'}\bar{\Gamma}_{n,k}H^{-1} &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{\sigma}_{ij,k}(\tilde{\lambda}_i\tilde{\lambda}_j' - H^{-1'}\lambda_i\lambda_j'H^{-1}) \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(\tilde{\sigma}_{ij,k} - \sigma_{ij,k})(\tilde{\lambda}_i\tilde{\lambda}_j' - H^{-1'}\lambda_i\lambda_j'H^{-1}) + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij,k}(\tilde{\lambda}_i\tilde{\lambda}_j' - H^{-1'}\lambda_i\lambda_j'H^{-1}) \\
&= I + II.
\end{aligned}
$$

Then, $I \to 0$ using the fact that it is zero mean process. We decompose the second term II as follows.

$$
II = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij,k}(\tilde{\lambda}_i - H^{-1}\lambda_i)\tilde{\lambda}_j' + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij,k}\lambda_i H^{-1}(\tilde{\lambda}_j - H^{-1'}\lambda_j)' = a + b.
$$

Then, we can show that $a \to 0$ and $b \to 0$ since a and b are of order $O_p(T_H^{-1/2}) + O_p(\delta_{NT_H}^{-2})$. Since $H \xrightarrow{p} H_0$, we can complete the proof. ■

The proof of Proposition 2.1 is straightforward by applying Theorem 2.2.

C Bootstrap procedure

In Algorithm 1, we present a description of our bootstrap procedure using AR-sieve + CSD bootstrap. In step 1, we resample the residuals of the factor model by AR sieve + CSD bootstrap. This is identical to the bootstrap method we present in the main text (Algorithm 1 in the main text). In step 2, we resample the regression residuals and obtain the bootstrap sample for the MIDAS regression model. In step 3, using the two-step estimation procedure, we can obtain the bootstrap estimators.

Algorithm 1 Bootstrap for the factor-MIDAS regression model

1. **Generate bootstrap factor model.** For $t_h = 1, \ldots, T_H$, let

$$X^*_{i,t_h} = \tilde{\lambda}'_i \tilde{f}_{t_h} + e^*_{i,t_h} \quad \text{and} \quad X^*_{t_h} = \tilde{\Lambda} \tilde{f}_{t_h} + e^*_{t_h},$$

where e^*_{i,t_h} is obtained as follows. For each $i = 1, \ldots, N$, select an order $p_i = p_i(T_H)$ with $p_i \ll T_H$ (e.g., by AIC) and fit a p_i-th order autoregressive model to $\tilde{e}_{i,1}, \ldots, \tilde{e}_{i,T_H}$, where $\tilde{e}_{i,t_h} = X_{i,t_h} - \tilde{\lambda}'_i \tilde{f}_{t_h}$.

Denote $\tilde{\phi}_i(p_i) = (\tilde{\phi}_{i,j}(p_i), j = 1, \ldots, p_i)$ as the Yule-Walker autoregressive parameter estimators, such that $\tilde{\phi}_i(p_i) = \tilde{\Gamma}(p_i)^{-1}\tilde{\gamma}_{p_i}$, with $\tilde{\gamma}_{p_i} = (\tilde{\gamma}_e(1), \tilde{\gamma}_e(2), \ldots, \tilde{\gamma}_e(p_i))'$ and $\tilde{\Gamma}(p_i) = (\tilde{\gamma}_e(r - s))_{r,s=1,2,\ldots,p_i}$ such that

$$\tilde{\gamma}_e(\tau) = \frac{1}{T_H} \sum_{t_h=1}^{T_H - |\tau|} (\tilde{e}_{i,t_h} - \bar{e}_i)(\tilde{e}_{i,t_h+|\tau|} - \bar{e}_i),$$

for $\tau = 0, \ldots, p_i$ and $\bar{e}_i = T_H^{-1} \sum_{t_h=1}^{T_H} \tilde{e}_{i,t_h}$. With chosen lag length $p_i = p_i(T_H)$, generate

$$e^*_{i,t_h} = \sum_{j=1}^{p_i} \tilde{\phi}_{i,j}(p_i) e^*_{i,t_h-j} + u^*_{i,t_h}, \quad t_h = 1, \ldots, T_H,$$

where $u^*_{t_h} = (u^*_{1,t_h}, \ldots, u^*_{N,t_h})' = \tilde{\Sigma}_u^{1/2} \eta_{t_h}$ with $\eta_{t_h} \sim$ i.i.d.$(0, I_N)$. Set initial conditions $e^*_{i,0}, \ldots, e^*_{i,1-p_i} = 0$ for $i = 1, \ldots, N$.

Choose $\tilde{\Sigma}_u = (\hat{\sigma}_{u,ij})_{i,j=1,\ldots,N}$ by thresholding, with

$$\hat{\sigma}_{u,ij} = \begin{cases} \tilde{\sigma}_{u,ij} & i = j \\ \tilde{\sigma}_{u,ij} \mathbb{1} \left(|\tilde{\sigma}_{u,ij}| > \omega \right) & i \neq j, \end{cases} \quad \text{where} \quad \tilde{\sigma}_{u,ij} = \frac{1}{T_H} \sum_{t_h=1}^{T_H} \tilde{u}_{i,t_h} \tilde{u}_{j,t_h},$$

ω is a threshold, and $\tilde{u}_{i,t_h} = \tilde{e}_{i,t_h} - \sum_{j=1}^{p_i} \tilde{\phi}_{i,j}(p_i) \tilde{e}_{i,t_h-j}$ for $i = 1, \ldots, N$ and $t_h = 1 + p_i, \ldots, T_H$.

2. **Generate bootstrap factor-MIDAS regression model.** For $t = 1, \ldots, T$, construct

$$y^*_t = \tilde{\beta}_0 + \tilde{\beta}'_1 \tilde{F}_t(\tilde{\theta}) + \varepsilon^*_t,$$

where $\varepsilon^*_t = \nu_t \hat{\varepsilon}_t$, $\hat{\varepsilon}_t = y_t - \tilde{\beta}_0 - \tilde{\beta}'_1 \tilde{F}_t(\tilde{\theta})$, and $\nu_t \sim$ i.i.d.$(0, 1)$ across t, independent of η_{t_h}.

3. **Extract bootstrap factors and estimate bootstrap parameters.** Obtain the estimated factors $\tilde{f}^*_{t_h}$ by principal component analysis on the bootstrap panel $X^*_{t_h}$. After, regress y^*_t on 1 and temporally aggregated factors $(\tilde{f}^{*\prime}_{t-1/m}, \ldots, \tilde{f}^{*\prime}_{t-K/m})'$ and obtain the bootstrap estimates $\tilde{\beta}^*$ and $\tilde{\theta}^*$.

D Proof of results in Section 3

In this section, we first deliver the bootstrap high-level conditions under which our bootstrap data generating process yields a consistent bootstrap distribution. Our bootstrap data generating process (DGP) is similar to the one proposed by GP (2014). Let $\{e^*_{t_h} = (e^*_{1,t_h}, \ldots, e^*_{N,t_h})'\}$ be a bootstrap sample from $\{\tilde{e}_{t_h} = (\tilde{e}_{1,t_h}, \ldots, \tilde{e}_{N,t_h})'\}$, where $\tilde{e}_{t_h} = X_{t_h} - \tilde{\Lambda}\tilde{f}_{t_h}$ are the residuals from the original panel dataset. $\{\varepsilon^*_t\}$ are the resampled bootstrap residuals from $\{\tilde{\varepsilon}_t = y_t - g(\tilde{F}_t; \tilde{\alpha})\}$. Using these two bootstrap samples, $\{e^*_{t_h}\}$ and $\{\varepsilon^*_t\}$, the bootstrap data generating process (DGP) is as follows.

$$X^*_{t_h} = \tilde{\Lambda}\tilde{f}_{t_h} + e^*_{t_h}, \text{ for } t_h = 1, \ldots, T_H,$$
$$y^*_t = \tilde{\beta}_0 + \tilde{\beta}_1'\tilde{F}_t(\tilde{\theta}) + \varepsilon^*_t, \text{ for } t = 1, \ldots, T.$$

We can obtain the bootstrap estimators by following a two-step process that is similar to the procedure used in the original sample: in the first step, we estimate the factors from a new bootstrap panel dataset $X^*_{t_h}$ and denote them by $\tilde{f}^*_{t_h}$, then in the second step, by regressing y^*_t on 1 and $\tilde{F}^*_t(\tilde{\theta})$, we can obtain the bootstrap estimators. We denote these estimators by $\tilde{\alpha}^*$, which are the analogues of NLS estimators from the original sample. Below conditions are our bootstrap high-level conditions. The conditions are similar to those of GP (2014).

Condition C.1* (*Weak Time Series and Cross Section Dependence in $e^*_{it_h}$*)

(a) $E^*(e^*_{i,t_h}) = 0$ *for all* (i, t_h).

(b) $\frac{1}{T_H}\sum_{t_h=1}^{T_H}\sum_{s_h=1}^{T_H} |\gamma^*_{s_h t_h}|^2 = O_p(1)$, *where* $\gamma^*_{s_h t_h} = E^*\left(\frac{1}{N}\sum_{i=1}^{N} e^*_{i,t_h} e^*_{i,s_h}\right)$.

(c) $\frac{1}{T_H^2}\sum_{t_h=1}^{T_H}\sum_{s_h=1}^{T_H} E^*\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(e^*_{i,t_h} e^*_{i,s_h} - E^*(e^*_{i,t_h} e^*_{i,s_h}))\right|^2 = O_p(1)$.

Condition C.2* (*Weak Dependence Among $\tilde{f}_{t_h}$, $\tilde{\lambda}_i$, and $\tilde{e}^*_{i,t_h}$*)

(a) $\frac{1}{T_H}\sum_{t_h=1}^{T_H}\sum_{s_h=1}^{T_H}\tilde{f}_{s_h}\tilde{f}_{t_h}'\gamma_{s_h t_h}^{*}=O_p(1)$.

(b) $\frac{1}{T_H}\sum_{t_h=1}^{T_H}E^{*}\left\|\frac{1}{\sqrt{T_H N}}\sum_{s_h=1}^{T_H}\sum_{i=1}^{N}\tilde{f}_{s_h}(e_{i,t_h}^{*}e_{i,s_h}^{*}-E^{*}(e_{i,t_h}^{*}e_{i,s_h}^{*}))\right\|^{2}=O_p(1)$.

(c) $E^{*}\left\|\frac{1}{\sqrt{T_H N}}\sum_{t_h=1}^{T_H}\sum_{i=1}^{N}\tilde{f}_{t_h}\tilde{\lambda}_i' e_{i,t_h}^{*}\right\|^{2}=O_p(1)$.

(d) $\frac{1}{T_H}\sum_{t_h=1}^{T_H}E^{*}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\tilde{\lambda}_i e_{i,t_h}^{*}\right\|^{2}=O_p(1)$.

(e) $\frac{1}{T_H}\sum_{t_h=1}^{T_H}\left(\frac{\tilde{\Lambda}' e_{t_h}^{*}}{\sqrt{N}}\right)\left(\frac{e_{t_h}^{*\prime}\tilde{\Lambda}}{\sqrt{N}}\right)-\tilde{\Gamma}=o_{p^*}(1)$, *in probability, where* $\tilde{\Gamma}\equiv\frac{1}{T_H}\sum_{t_h=1}^{T_H}Var^{*}\left(\frac{1}{\sqrt{N}}\tilde{\Lambda}' e_{t_h}^{*}\right)>0$.

Condition C.3* (*Serial Dependence among $\tilde{f}_{t_h}$, $\tilde{\lambda}_i$, and $\tilde{e}_{i,t_h}^{*}$*)

(a) $\frac{1}{T_H}\sum_{t_h=1}^{T_H}E^{*}\left\|\frac{1}{\sqrt{T_H N}}\sum_{s_h=1}^{T_H}\sum_{i=1}^{N}\tilde{f}_{s_h}(e_{i,t_h}^{*}e_{i,s_h-k}^{*}-E^{*}(e_{i,t_h}^{*}e_{i,s_h-k}^{*}))\right\|^{2}=O_p(1)$ *for all* k.

(b) $E^{*}\left\|\frac{1}{\sqrt{T_H N}}\sum_{t_h=1}^{T_H}\tilde{f}_{t_h}e_{t_h-k}^{*\prime}\tilde{\Lambda}\right\|^{2}=O_p(1)$ *for all* k.

(c) $\frac{1}{T_H}\sum_{t_h=1}^{T_H}\left(\frac{\tilde{\Lambda} e_{t_h}^{*}}{\sqrt{N}}\right)\left(\frac{e_{t_h-k}^{*\prime}\tilde{\Lambda}}{\sqrt{N}}\right)-\tilde{\Gamma}_k=o_{p^*}(1)$, *in probability, where* $\tilde{\Gamma}_k\equiv\frac{1}{T_H}\sum_{t_h=1}^{T_H}Cov^{*}\left(\frac{\tilde{\Lambda}' e_{t_h}^{*}}{\sqrt{N}},\frac{\tilde{\Lambda}' e_{t_h-k}^{*}}{\sqrt{N}}\right)>0$.

Condition C.4* (*Weak Dependence Between e_{i,t_h}^{*} and ε_t^{*}*)

(a) $\frac{1}{T}\sum_{t=1}^{T}E^{*}\left|\frac{1}{\sqrt{TN}}\sum_{s=1}^{T}\sum_{i=1}^{N}\varepsilon_s^{*}(e_{i,t-j/m}^{*}e_{i,s-j/m}^{*}-E^{*}(e_{i,t-j/m}^{*}e_{i,s-j/m}^{*}))\right|^{2}=O_p(1)$ *for* $j=0,\ldots,m-1$.

(b) $E^{*}\left\|\frac{1}{\sqrt{TN}}\sum_{t=1}^{T}\sum_{i=1}^{N}\tilde{\lambda}_i e_{i,t-j/m}^{*}\varepsilon_t^{*}\right\|^{2}=O_p(1)$, *where* $E(e_{i,t-j/m}^{*})=0$ *for all* (i,t) *and* $j=0,\ldots,m-1$.

Condition C.5* (*Bootstrap CLT*)

(a) $E^{*}(\varepsilon_t^{*})=0$ *and* $\frac{1}{T}\sum_{t=1}^{T}E^{*}\left|\varepsilon_t\right|^{2}=O_p(1)$.

(b) $\tilde{\Omega}^{-1/2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{g}_{\alpha,t}\varepsilon_t^* \xrightarrow{d*} N(0, I_{r+p})$*, in probability, where* $E^*\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{g}_{\alpha,t}\varepsilon_t^*\right\|^2 = O_p(1)$ *and* $\tilde{g}_{\alpha,t} = \partial g(\tilde{F}_t, \alpha)/\partial\alpha$*, and* $\tilde{\Omega} \equiv Var^*\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{g}_{\alpha,t}\varepsilon_t^*\right) > 0$.

Condition C.6* (*Bootstrap Consistency*)

(a) $\text{plim}\,\tilde{\Omega} = \Phi_0\Omega\Phi_0'$*, where* $\tilde{\Omega} = Var^*\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{g}_{\alpha,t}\varepsilon_t^*\right)$ *and* $\tilde{g}_{\alpha,t} \equiv \partial g(\tilde{F}_t, \alpha)/\partial\alpha$.

(b) $\text{plim}\,\tilde{\Gamma} = H_0\Gamma H_0'$ *and* $\text{plim}\,\tilde{\Gamma}_{j-l} = H_0\Gamma_{j-l}H_0'$.

Conditions C.1* through C.4* are the bootstrap analogues of Assumptions A.1 to A.6 in Appendix A. Conditions C.1*-C.2* are similar to the bootstrap high level conditions in GP (2014). The mean of bootstrap residuals are required to be zeros for all (i, t_h), which implies that we need to recenter the residuals when we resample them. Condition C.3* is a new set of high-level conditions required in our context. Unlike in GP (2014), since our bias contains the term which relies on serial dependence in the idiosyncratic error term in the factor model, we impose weak serial dependence among $\tilde{f}_{t_h}$, $\tilde{\lambda}_i$ and e^*_{i,t_h} in Condition C.3*. Note that since $\tilde{f}_{t_h}$ and $\tilde{\lambda}_i$ are fixed in the bootstrap world, serial dependence in the factors can be implied by restricting the serial dependence of e_{i,t_h}. Condition C.4* is similar to Condition C* in GP (2014), and we restrict the dependence between two bootstrap residuals. Condition C.5* implies that we can apply a central limit theorem on the score vector, $\tilde{g}_{\alpha,t}\varepsilon_t^*$. In Condition C.6*, we provide conditions for consistency of the bootstrap distribution. In Condition C.6*-(a), Ω denotes the bootstrap variance of the score vector in the bootstrap world and it is a bootstrap analogue of Ω. It implies that the bootstrap variance is rotated with a block diagonal matrix, Φ_0. This is because the score vector $\tilde{g}_{\alpha,t} = \left(F_t'(\theta)H', \beta'\frac{\partial F_t(\theta)}{\partial\theta'}\right)'$ is a rotated version of $g_{\alpha,t}$, where the rotation is given by Φ_0. Similarly, $\tilde{\Gamma}$ and $\tilde{\Gamma}_{j-l}$, defined in Condition C.2* and Condition C.3* are the bootstrap analogues of Γ and Γ_{j-l}, respectively.

Condition C.6*-(a) and (b) imply that it is crucial how we mimic the error terms of the MIDAS regression and the idiosyncratic factor error terms in the bootstrap world. Moreover, in our context, since the bias depends on both serial and cross-sectional dependence of e_{t_h}, the idiosyncratic error term in the bootstrap world should mimic the dependence in the time series and cross-sectional dimension.

Remark 1 *Note that $\tilde{\alpha}^*$ is obtained by regressing y_t^* on 1 and a temporally aggregated version of the lags of the bootstrap estimated factors, $\tilde{F}_t^*(\tilde{\theta})$. The bootstrap estimated factors, $\tilde{f}_{t_h}^*$, consistently estimate the rotated version of true "latent" bootstrap factors, $H^*\tilde{f}_{t_h}$, where $H^* = \tilde{V}^{*-1}\frac{\tilde{f}^{*\prime}\tilde{f}}{T_H}\frac{\tilde{\Lambda}'\tilde{\Lambda}}{N}$ and $\tilde{V}^*$ is the $r \times r$ diagonal matrix containing on the main diagonal the r largest eigenvalues of $X^*X^{*\prime}/NT_H$, in decreasing order. This matrix is the bootstrap analogue of the rotation matrix in the original sample, $H = \tilde{V}^{-1}\frac{\tilde{f}'f}{T_H}\frac{\Lambda'\Lambda}{N}$. As discussed in GP (2014), the indeterminacy of the rotation matrix is not a problem in the bootstrap world, as H^* does not depend on the population values. Moreover, H^* is asymptotically equal to $H_0^* = \mathrm{diag}(\pm 1)$, where the sign is determined by the sign of $\tilde{f}^{*\prime}\tilde{f}/T_H$. This implies that the bootstrap factors are identified up to a change of sign.*

Remark 2 *Similar to the discussion in GP (2014) regarding the rotation of the bootstrap estimators, our NLS estimators of bootstrap DGP rotate due to the rotation in the factors in the bootstrap world. Note that we can rewrite y_t^* as follows.*

$$y_t^* = \tilde{\beta}_0 + \tilde{\beta}_1' H^{*-1}\tilde{F}_t^*(\tilde{\theta}) + \tilde{\beta}_1' H^{*-1}(H^*\tilde{F}_t(\tilde{\theta}) - \tilde{F}_t^*(\tilde{\theta})) + \varepsilon_t^* = g(\tilde{F}_t^*, \tilde{\alpha}) + \xi_t^*,$$

where $g(\tilde{F}_t^*, \tilde{\alpha}) \equiv \tilde{\beta}_0 + \tilde{\beta}_1' H^{*-1}\tilde{F}_t^*(\tilde{\theta})$ and $\xi_t \equiv \tilde{\beta}_1' H^{*-1}(H^*\tilde{F}_t(\tilde{\theta}) - \tilde{F}_t^*(\tilde{\theta})) + \varepsilon_t^*$. Thus, $\tilde{\alpha}^*$ estimates $(\Phi^*)^{-1}\tilde{\alpha}$, where $\Phi^* = diag(1, H^*, I_p)$ is a block diagonal matrix. $(\Phi^*)^{-1}\tilde{\alpha}$ are the rotated version of NLS estimators in the original sample. As H^* is asymptotically equal *to* H_0^*, $(\Phi^*)^{-1}\tilde{\alpha}$ *is equal to* $(\Phi_0^*)^{-1}\tilde{\alpha}$, *where* $\Phi_0^* = \text{diag}(1, H_0^*, I_p)$, *and* $(\Phi_0^*)^{-1}\tilde{\alpha}$ *is the sign-adjusted version of* $\tilde{\alpha}$.

Lemma D.1 *Let the Assumptions A.1-A.5 in Appendix A hold and consider any residual-based bootstrap scheme for which Conditions C.1*-C.5* are verified. Suppose* $\sqrt{T}/N \to c,\ 0 \le c < \infty$. *In addition, let the two following conditions hold: (1) Condition C.6*-(a) is verified and (2)* $c = 0$ *or Condition C.6*-(b) is verified; then as* $N, T \to \infty$,

$$\sqrt{T}(\tilde{\alpha}^* - (\Phi_0^*)^{-1}\tilde{\alpha}) \xrightarrow{d^*} N(-c(\Phi_0^*)^{-1}\Delta_\alpha, (\Phi_0^*)^{-1}\Sigma_\alpha(\Phi_0^*)^{-1}),$$

in probability and Δ_α *and* Σ_α *are defined in Theorem 2.1.*

Remark 3 *In Lemma D.1, we derive the bootstrap distribution of the estimators,* $\tilde{\alpha}^*$. *According to Lemma D.1, the distribution of* $\sqrt{T}(\tilde{\alpha}^* - (\Phi_0^*)^{-1}\tilde{\alpha})$ *follows a normal distribution with a non-zero mean vector,* $-c(\Phi_0^*)^{-1}\Delta_\alpha$. *The asymptotic bias is proportional to* $(H_0^*)^{-1}\tilde{\beta}$. *However, the weighting parameters* $\tilde{\theta}^*$ *are not affected by the rotation problem.*

Remark 4 *To match the bootstrap distribution with the limiting distribution of the estimators in the original sample to achieve bootstrap consistency since our rotation matrix* H^* *may not be an identity matrix. Therefore, we consider the rotated version of our bootstrap results, given by* $\sqrt{T}(\Phi^*\tilde{\alpha}^* - \tilde{\alpha})$. *For the consistency of the rotated bootstrap results, we rely on the Corollary 3.1. in GP (2014) such that* $\sup_{x \in \mathbb{R}^{r+p}} |P^*(\sqrt{T}(\Phi_0^*\tilde{\alpha}^* - \tilde{\alpha}) \le x) - P(\sqrt{T}(\tilde{\alpha} - \alpha) \le x)| \xrightarrow{p} 0$. *For detail, see GP (2014). This corollary justifies the use of a residual-based bootstrap method in the context of the factor-MIDAS regression models.*

Notation: P^* denotes the bootstrap probability measure, conditional on the original sample. The bootstrap measure P^* depends on the original sample size N, T and T_H, and sample realization ω, but for a simpler notation, we omit these and write P^* for $P^*_{NT,\omega}$. We write $T^*_{NT} = o_{p^*}(1)$, in probability, or $T^*_{NT} \xrightarrow{p^*} 0$, in probability, for any bootstrap test statistics T^*_{NT}, if, when for any $\delta > 0$, $P^*(|T^*_{NT}| > \delta) = o_p(1)$. If for all $\delta > 0$, there exists $M_\delta < \infty$ such that $\lim_{N,T\to\infty} P[P^*(|T^*_{NT}| > M_\delta) > \delta] = 0$, we write as $T^*_{NT} = O_{p^*}(1)$, in probability. We write $T^*_{NT} \xrightarrow{d*} D$, in probability, if T^*_{NT} weakly converges to the distribution D under P^*, conditional on a sample with probability that converges to one, i.e. $E^*(f(T^*_{NT})) \xrightarrow{p} E(f(D))$ for all bounded and uniformly continuous function f.

Lemma D.2 $\frac{1}{T}\sum_{t=1}^{T} \varepsilon^*_t(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m}) = o_{p^*}(1)$.

Lemma D.3 *If* $\sqrt{T}/N \to c$, *where* $0 \leq c < \infty$,

(a) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})' = \frac{\sqrt{T}}{N}\tilde{V}^{*-1}H^*\Gamma^*H^*\tilde{V}^{*-1} + o_{p^*}(1)$,

(b) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})(\tilde{f}^*_{t-l/m} - H^*\tilde{f}_{t-l/m})' = \frac{\sqrt{T}}{N}\tilde{V}^{*-1}H^*\Gamma^*_{j-l}H^*\tilde{V}^{*-1} + o_{p^*}(1)$,

(c) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{f}_{t-j/m}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})' = \frac{\sqrt{T}}{N}H^*\Gamma^*\left(\frac{1}{T_H}\sum_{s_h=1}^{T_H}\tilde{f}_{s_h}\tilde{f}^{*\prime}_{s_h}\right)\tilde{V}^{*-2} + o_{p^*}(1)$,

(d) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{f}_{t-l/m}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})' = \frac{\sqrt{T}}{N}H^*\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-l/m}\tilde{f}'_{t-j/m}\right)\Gamma^*\left(\frac{1}{T_H}\sum_{s_h=1}^{T_H}\tilde{f}_{s_h}\tilde{f}^{*\prime}_{s_h}\right)\tilde{V}^{*-2} +$
$o_{p^*}(1)$.

Lemma D.4 *If* $\sqrt{T}/N \to c$, *where* $0 \leq c < \infty$,

(a) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{F}^*_t(\tilde{\theta}) - H^*\tilde{F}_t(\tilde{\theta}))(\tilde{F}^*_t(\tilde{\theta}) - H^*\tilde{F}_t(\tilde{\theta}))'$
$= cH^*_0\tilde{V}^{-1}\left(\sum_{j=1}^{K} w_j(\tilde{\theta})\Gamma^* w_j(\tilde{\theta}) + \sum_{j=1}^{K} w_j(\tilde{\theta})\Gamma^*_{j-l}w_l(\tilde{\theta})\right)\tilde{V}^{-1}H^*_0 + o_{p^*}(1)$,

(b) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{F}_t(\tilde{\theta})(\tilde{F}^*_t(\tilde{\theta}) - H^*\tilde{F}_t(\tilde{\theta}))'$
$= cH^*_0\left[\sum_{j=1}^{K} w^2_j(\tilde{\theta}) + \sum_{j=1}^{K}\sum_{l\neq j}^{K} w_j(\tilde{\theta})\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-l/m}\tilde{f}'_{t-j/m}\right)w_l(\tilde{\theta})\right]\Gamma^*\tilde{V}^{-2}H^*_0 + o_{p^*}(1)$.

Proof of Lemma D.1. Since in the bootstrap world, $\tilde{\alpha}^*$ maximizes the following objective function:

$$\tilde{Q}_T^*(\tilde{\alpha}) = -\frac{1}{T}\sum_{t=1}^{T}[y_t - g(\tilde{F}_t^*, \tilde{\alpha})]^2.$$

where $g(\tilde{F}_t^*, \tilde{\alpha}) = \tilde{\beta}' H^{*-1}\tilde{F}_t^*(\tilde{\theta})$. Then, we have

$$\sqrt{T}(\tilde{\alpha}^* - (\Phi^*)^{-1}\tilde{\alpha}) = -\left[\frac{1}{T}\sum_{t=1}^{T} H(\tilde{F}_t^*, \tilde{\alpha}_T)\right]^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} s(\tilde{F}_t^*, \tilde{\alpha}),$$

where $s(\tilde{F}_t^*, \tilde{\alpha})$ is a score vector and $H(\tilde{F}_t^*, \tilde{\alpha})$ is a Hessian matrix in the bootstrap world. $\tilde{\alpha}_T$ is intermediate between $\tilde{\alpha}$ and $\tilde{\alpha}^*$. We analyse each term. We can write the score vector as follows.

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} s(\tilde{F}_t^*, \tilde{\alpha}) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}[\varepsilon_t^* + \tilde{\beta}' H^{*-1}(H^*\tilde{F}_t(\tilde{\theta}) - \tilde{F}_t^*(\tilde{\theta}))]\frac{\partial g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha}},$$

where the partial derivative is

$$\frac{\partial g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha}} = \Phi^*\frac{\partial g(\tilde{F}_t, \alpha)}{\partial \alpha} + P_t^*, \text{ where } P_t^* = \begin{bmatrix} 0 \\ \tilde{F}_t^*(\tilde{\theta}) - H^*\tilde{F}_t(\theta) \\ \left(\frac{\partial \tilde{F}_t^*(\tilde{\theta})'}{\partial \tilde{\theta}} H^{*-1'}\tilde{\beta} - \frac{\partial \tilde{F}_t(\theta)'}{\partial \theta} H^{-1'}\beta\right) \end{bmatrix},$$

and $\Phi^* = diag(1, H^*, I_p)$. Under this decomposition, we can analyse $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\frac{\partial g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha}}$ into two non-zero blocks of P_t^*. The second block can be written as follows.

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*(\tilde{F}_t^*(\tilde{\theta}) - H^*\tilde{F}_t(\theta)) \\
&\quad = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\left[\sum_{j=1}^{K} w_j(\tilde{\theta})(\tilde{f}_{t-j/m}^* - H^*\tilde{f}_{t-j/m}) + \sum_{j=1}^{K}(w_j(\tilde{\theta}) - w_j(\theta))H^*\tilde{f}_{t-j/m}\right] \\
&\quad = \sum_{j=1}^{K} w_j(\tilde{\theta})\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*(\tilde{f}_{t-j/m}^* - H^*\tilde{f}_{t-j/m}) + \sum_{j=1}^{K}(w_j(\tilde{\theta}) - w_j(\theta))H^*\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\tilde{f}_{t-j/m} \\
&\quad = o_{p^*}(1).
\end{aligned}$$

Since $\tilde{\theta} \xrightarrow{p} \theta$ and weighting function is continuous function, we can use continuous mapping theorem and have the second part as $o_p(1)$. By Lemma D.2 we can show that the first part is $o_{p^*}(1)$. The third part can be argued similarly. Since it is easier to check for each row, we write k-th row of the third block in P_t^* as $\big(\frac{\partial \tilde{F}_{k,t}^*(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} H_k^{*-1'}\tilde{\beta}_k - \frac{\partial \tilde{F}_{k,t}(\theta_k)}{\partial \theta_k} H_k^{-1'}\beta_k\big)$. Then, for this k-th row, we can write it as follows.

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\left(\frac{\partial \tilde{F}_{k,t}^*(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} H_k^{*-1'}\tilde{\beta}_k - \frac{\partial \tilde{F}_{k,t}(\theta_k)}{\partial \theta_k} H_k^{-1'}\beta_k\right) \\
&= H_k^{*-1'}\tilde{\beta}_k\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\sum_{j=1}^{K}\frac{\partial w_{j,k}(\tilde{\theta}_k)}{\partial \tilde{\theta}_k}(\tilde{f}_{k,t-j/m}^* - H_k^*\tilde{f}_{k,t-j/m})\right. \\
&\qquad \left. + \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\sum_{j=1}^{K}\left\{\frac{\partial w_{j,k}(\tilde{\theta}_k)}{\partial \tilde{\theta}_k} - \frac{\partial w_{j,k}(\theta_k)}{\partial \theta_k}\right\}\tilde{f}_{k,t-j/m}\right] \\
&\qquad + (\tilde{\beta}_k - H_k^{-1'}\beta_k)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\left[\sum_{j=1}^{K}\frac{\partial w_{j,k}(\theta_k)}{\partial \theta_k}\tilde{f}_{k,t-j/m}\right] \\
&= o_{p^*}(1),
\end{aligned}$$

where H_k is the k-th diagonal element in the rotation matrix H and β_k is the k-th slope parameter in β. We can obtain the second equality because $\tilde{\beta} \xrightarrow{p} H^{-1'}\beta$ and Lemma D.2.

Finally, we have the following result.

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\frac{\partial g(\tilde{F}_t;\tilde{\alpha})}{\partial\tilde{\alpha}}\xrightarrow{d^*}N(0,\Phi_0^*\tilde{\Omega}\Phi_0^*), \tag{5}$$

where $\Phi_0^* = \text{plim}\,\Phi^*$, $\tilde{\Omega}\equiv\text{Var}^*\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t^*\tilde{g}_{\alpha,t}\right)$, and $\tilde{g}_{\alpha,t}=\partial g(\tilde{F}_t,\alpha)/\partial\alpha$.

Now, we analyse the second term in the score vector $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{\beta}'H^{*-1}(H^*\tilde{F}_t(\tilde{\theta})-\tilde{F}_t^*(\tilde{\theta}))\frac{\partial g(\tilde{F}_t^*,\alpha)}{\partial\tilde{\alpha}}$ with respect to $\tilde{\beta}$ and $\tilde{\theta}$, respectively. (Note that there is no bias with respect to $\tilde{\beta}_0$, therefore we focus on $\tilde{\beta}_1$ here.) By Lemma D.4, the score vector with respect to $\tilde{\beta}_1$ can be written as follows.

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(H^*\tilde{F}_t(\tilde{\theta})-\tilde{F}_t^*(\tilde{\theta}))\tilde{F}_t^*(\tilde{\theta})'H^{*-1\prime}\tilde{\beta}_1\\
&=-\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{F}_t^*(\tilde{\theta})-H^*\tilde{F}_t(\tilde{\theta}))(\tilde{F}_t^*(\tilde{\theta})-H^*\tilde{F}_t(\tilde{\theta}))'+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}H^*\tilde{F}_t(\tilde{\theta})(\tilde{F}_t^*(\tilde{\theta})-H^*\tilde{F}_t(\tilde{\theta}))'\right]H^{*-1\prime}\tilde{\beta}_1\\
&=-cH_0^*\left[\tilde{V}^{-1}\left\{\sum_{j=1}^{K}w_j(\tilde{\theta})\tilde{\Gamma}w_j(\tilde{\theta})+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\tilde{\theta})\tilde{\Gamma}_{j-l}w_l(\tilde{\theta})\right\}\tilde{V}^{-1}\right.\\
&\quad\left.+\left\{\sum_{j=1}^{K}w_j^2(\tilde{\theta})+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\tilde{\theta})\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{F}_{t-j/m}\tilde{F}_{t-l/m}'\right)w_l(\tilde{\theta})\right\}\tilde{\Gamma}\tilde{V}^{-2}\right]\tilde{\beta}_1\\
&=-cH_0^*\tilde{B}_{\beta_1}+o_{p^*}(1)
\end{aligned}$$

in probability, where we define $\tilde{B}_{\beta_1}$ as follows.

$$\begin{aligned}
\tilde{B}_{\beta_1}\equiv&\left[\tilde{V}^{-1}\left\{\sum_{j=1}^{K}w_j(\tilde{\theta})\tilde{\Gamma}w_j(\tilde{\theta})+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\tilde{\theta})\tilde{\Gamma}_{j-l}w_l(\tilde{\theta})\right\}\tilde{V}^{-1}\right.\\
&\left.+\left\{\sum_{j=1}^{K}w_j^2(\tilde{\theta})+\sum_{j=1}^{K}\sum_{l\neq j}^{K}w_j(\tilde{\theta})\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{F}_{t-j/m}\tilde{F}_{t-l/m}'\right)w_l(\tilde{\theta})\right\}\tilde{\Gamma}\tilde{V}^{-2}\right]\tilde{\beta}_1.
\end{aligned}$$

We can also rewrite the part with respect to $\tilde{\theta}$ by Lemma D.4 as follows.

$$
\begin{aligned}
&\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\partial \tilde{F}_t^*(\tilde{\theta})}{\partial \tilde{\theta}} H^{*-1'} \tilde{\beta}_1 \tilde{\beta}_1' H^{*-1} [H^* \tilde{F}_t(\tilde{\theta}) - \tilde{F}_t^*(\tilde{\theta})] \\
&\quad = -c\tilde{\beta}_1 \circ \left[\tilde{V}^{-1} \left\{ \sum_{j=1}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} \tilde{\Gamma} w_j(\tilde{\theta}) + \sum_{j=1}^{K} \sum_{l \neq j}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} \tilde{\Gamma}_{j-l} w_l(\tilde{\theta}) \right\} \tilde{V}^{-1} \right. \\
&\qquad \left. + \left\{ \sum_{j=1}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} w_j(\tilde{\theta}) + \sum_{j=1}^{K} \sum_{l \neq j}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} \left(\frac{1}{T} \sum_{t=1}^{T} \tilde{F}_{t-j/m} \tilde{F}_{t-l/m}' \right) w_l(\tilde{\theta}) \right\} \tilde{\Gamma} \tilde{V}^{-2} \right] \tilde{\beta}_1 \\
&\quad = -c\tilde{B}_\theta + o_{p^*}(1),
\end{aligned}
$$

in probability, where we define $\tilde{B}_\theta$ as follows.

$$
\begin{aligned}
\tilde{B}_\theta \equiv \tilde{\beta}_1 \circ &\left[\tilde{V}^{-1} \left\{ \sum_{j=1}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} \tilde{\Gamma} w_j(\tilde{\theta}) + \sum_{j=1}^{K} \sum_{l \neq j}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} \tilde{\Gamma}_{j-l} w_l(\tilde{\theta}) \right\} \tilde{V}^{-1} \right. \\
&\left. + \left\{ \sum_{j=1}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} w_j(\tilde{\theta}) + \sum_{j=1}^{K} \sum_{l \neq j}^{K} \frac{\partial w_j(\tilde{\theta})}{\partial \tilde{\theta}} \left(\frac{1}{T} \sum_{t=1}^{T} \tilde{F}_{t-j/m} \tilde{F}_{t-l/m}' \right) w_l(\tilde{\theta}) \right\} \tilde{\Gamma} \tilde{V}^{-2} \right] \tilde{\beta}_1.
\end{aligned}
$$

Next, we derive the hessian matrix. We first rewrite it as follows.

$$
\frac{1}{T} \sum_{t=1}^{T} H(\tilde{F}_t^*, \tilde{\alpha}) = \frac{1}{T} \sum_{t=1}^{T} \xi_t \frac{\partial^2 g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha} \partial \tilde{\alpha}'} + \frac{1}{T} \sum_{t=1}^{T} \frac{\partial g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha}} \frac{\partial g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha}'} = H_1 + H_2.
$$

Then, H_1 is $o_{p^*}(1)$ by Condition C.5*-(b) and the results in the proof for Lemma D.3. The second term H_2 converges in probability to $\Phi_0^* \tilde{\Sigma} \Phi_0^*$ as following:

$$
\frac{1}{T} \sum_{t=1}^{T} \frac{\partial g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha}} \frac{\partial g(\tilde{F}_t^*, \tilde{\alpha})}{\partial \tilde{\alpha}'} \xrightarrow{p^*} \Phi_0^* E\Big[\frac{\partial g(\tilde{F}_t, \alpha)}{\partial \alpha} \frac{\partial g(\tilde{F}_t, \alpha)}{\partial \alpha'} \Big] \Phi_0^* \equiv \Phi_0^* \tilde{\Sigma} \Phi_0^*, \tag{6}
$$

where $E\Big[\frac{\partial g(\tilde{F}_t,\alpha)}{\partial \alpha} \frac{\partial g(\tilde{F}_t,\alpha)}{\partial \alpha'}\Big] \equiv \tilde{\Sigma}$. We can obtain this by rewriting $\frac{\partial g(\tilde{F}_t^*,\tilde{\alpha})}{\partial \tilde{\alpha}} = \Phi^* \frac{\partial g(\tilde{F}_t,\alpha)}{\partial \alpha} + P_t^*$. Then, $\frac{1}{T}\sum_{t=1}^{T} \frac{\partial g(\tilde{F}_t,\tilde{\alpha})}{\partial \tilde{\alpha}} P_t^{*'} = o_{p^*}(1)$ and $\frac{1}{T}\sum_{t=1}^{T} P_t^* P_t^{*'} = o_{p^*}(1)$, in probability. By putting all together, we have

$$
\sqrt{T}(\tilde{\alpha}^* - (\Phi_0^*)^{-1}\tilde{\alpha}) \xrightarrow{d^*} N(-c(\Phi_0^* \tilde{\Sigma} \Phi_0^*)^{-1} \Phi_0^* \tilde{B}_\alpha, \Phi_0^{*-1} \tilde{\Sigma}^{-1} \tilde{\Omega} \tilde{\Sigma}^{-1} \Phi_0^{*-1}), \tag{7}
$$

in probability, where $\tilde{B}_\alpha = (0, \tilde{B}_{\beta_1}, \tilde{B}_\theta)'$. Under Assumptions A.1-A.6, we have $\text{plim}\, \tilde{V} = V$, $\text{plim}\, \tilde{\alpha} = \Phi^{-1}\alpha$, $\text{plim}\, \Phi^* = \Phi_0^*$, and $\text{plim}\, \tilde{\Omega} = \Phi_0 \Omega \Phi_0$. This implies that $\sqrt{T}(\tilde{\alpha}^* - (\Phi_0^*)^{-1}\tilde{\alpha}) \xrightarrow{d^*} N(-c\Phi_0^{*-1}\Delta_\alpha, \Phi_0^{*-1}\Sigma_\alpha \Phi_0^{*-1})$, in probability. ■

The proof of Lemma D.3 is similar to the proof of Lemma B.2 in GP (2014) and Lemma D.3 - (a) and (c) are similar to the proof of Lemma B.3 - (a) and (b) in GP (2014), respectively. Thus, we focus here to prove Lemma D.3-(b) and (d), which are new.

Proof of Lemma D.3. Part(b): Using the identity in GP (2014), we can rewrite the part (b) as follows.

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})(\tilde{f}^*_{t-l/m} - H^*\tilde{f}_{t-l/m})' \\
&= \tilde{V}^{*-1}\frac{1}{T}\sum_{t=1}^{T}(A^*_{1,t-j/m} + A^*_{2,t-j/m} + A^*_{3,t-j/m} + A^*_{4,t-j/m}) \\
&\quad \times (A^*_{1,t-l/m} + A^*_{2,t-l/m} + A^*_{3,t-l/m} + A^*_{4,t-l/m})'\tilde{V}^{*-1}.
\end{aligned}
$$

Ignoring $\tilde{V}^{*-1} = O_{p^*}(1)$, we can show that the terms except $\frac{1}{T}\sum_{t=1}^{T} A^*_{3,t-j/m}A^{*\prime}_{3,t-l/m}$ are negligible. For example, we have $\frac{1}{T}\sum_{t=1}^{T} A^*_{1,t-j/m}A^{*\prime}_{1,t-l/m} = O_{p^*}(T^{-1})$, $\frac{1}{T}\sum_{t=1}^{T} A^*_{2,t-j/m}A^{*\prime}_{2,t-l/m} = O_{p^*}(N^{-1}\delta_{NT_H}^{-2})$, and $\frac{1}{T}\sum_{t=1}^{T} A^*_{4,t-j/m}A^{*\prime}_{4,t-l/m} = O_{p^*}(N^{-1}\delta_{NT_H}^{-2})$. The cross terms are: $\frac{1}{T}\sum_{t=1}^{T} A^*_{1,t-j/m}A^{*\prime}_{2,t-l/m}$ $O_{p^*}(T^{-1/2}N^{-1/2}\delta_{NT_H}^{-1})$, $\frac{1}{T}\sum_{t=1}^{T} A^*_{1,t-j/m}A^{*\prime}_{3,t-l/m} = O_{p^*}(T^{-1/2}N^{-1/2})$, $\frac{1}{T}\sum_{t=1}^{T} A^*_{1,t-j/m}A^{*\prime}_{4,t-l/m} = O_{p^*}(T^{-1/2}N^{-1/2})$, $\frac{1}{T}\sum_{t=1}^{T} A^*_{2,t-j/m}A^{*\prime}_{3,t-l/m} = O_{p^*}(N^{-1}\delta_{NT_H}^{-2})$, $A^*_{2,t-j/m}A^{*\prime}_{4,t-l/m} = O_{p^*}(N^{-1}\delta_{NT_H}^{-1})$, and $\frac{1}{T}\sum_{t=1}^{T} A^*_{3,t-j/m}A^{*\prime}_{4,t-l/m} = O_{p^*}(N^{-1}\delta_{NT_H}^{-1})$. Since we can show that

$$
\frac{1}{T}\sum_{t=1}^{T} A^*_{3,t-j/m}A^{*\prime}_{3,t-l/m} = \frac{1}{N}H^*\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\tilde{\Lambda}' e^*_{t-j/m}}{\sqrt{N}}\right)\left(\frac{e^{*\prime}_{t-l/m}\tilde{\Lambda}}{\sqrt{N}}\right)H^* + o_{p^*}(1),
$$

we have

$$
\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})(\tilde{f}^*_{t-l/m} - H^*\tilde{f}_{t-l/m})' = \frac{\sqrt{T}}{N}\tilde{V}^{*-1}H^*\tilde{\Gamma}_{j-l}H^*\tilde{V}^{*-1} + o_{p^*}(1),
$$

where we define $\Gamma^*_{j-l} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\tilde{\Lambda}' e^*_{t-j/m}}{\sqrt{N}}\right)\left(\frac{e^{*\prime}_{t-l/m}\tilde{\Lambda}}{\sqrt{N}}\right)$. Part (d): Similar to the identity we used in part (b), we can rewrite part (d) as follows.

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{f}_{t-j/m}(\tilde{f}^*_{t-l/m} - H^*\tilde{f}_{t-l/m})' \\
&\quad = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{f}_{t-j/m}(A^*_{1,t-l/m} + A^*_{2,t-l/m} + A^*_{3,t-l/m} + A^*_{4,t-l/m})'\tilde{V}^{*-1} \\
&\quad \equiv \sqrt{T}H^*(d^*_{f1} + d^*_{f2} + d^*_{f3} + d^*_{f4})'\tilde{V}^{*-1},
\end{aligned}$$

where $d^*_{fi} \equiv \frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-j/m}A^{*\prime}_{i,t-l/m}$ for $i = 1, 2, 3, 4$. Then, we can obtain $d^*_{f1} = O_{p^*}(\delta^{-1}_{NT_H}T^{-1/2}) + O_{p^*}(T_H^{-1})$, $d^*_{f2} = O_{p^*}((TN)^{-1/2})$ by Condition C.3*-(a) and $d^*_{f3} = O_{p^*}((TN)^{-1/2})$ by Condition C.3*-(b). Finally, $d^*_{f4} = \frac{1}{N}\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-l/m}\tilde{F}'_{t-j/m}\right)\Gamma^*\left(\frac{1}{T_H}\sum_{t=1}^{T_H}\tilde{f}_t\tilde{f}^{*\prime}_t\right)\tilde{V}^{*-1} + o_{p^*}(1)$. Thus,

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{f}_{t-j/m}(\tilde{f}^*_{t-l/m} - H^*\tilde{f}_{t-l/m})' \\
&\quad = \frac{\sqrt{T}}{N}H^*\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-l/m}\tilde{f}'_{t-j/m}\right)\Gamma^*\left(\frac{1}{T_H}\sum_{s=1}^{T_H}\tilde{f}_s\tilde{f}^{*\prime}_s\right)\tilde{V}^{*-2} + o_{p^*}(1).
\end{aligned}$$

■

Proof of Lemma D.4. Part (a): We rewrite part (a) and apply Lemma D.3.

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\sum_{j=1}^{K} w_j(\tilde{\theta})(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})\right]\left[\sum_{j=1}^{K} w_j(\tilde{\theta})(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})\right]'$$
$$= \sum_{j=1}^{K} w_j(\tilde{\theta})\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})'\right] w_j(\tilde{\theta})$$
$$+ \sum_{j=1}^{K}\sum_{l\neq j}^{K} w_j(\tilde{\theta})\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})(\tilde{f}^*_{t-l/m} - H^*\tilde{f}_{t-l/m})'\right] w_l(\tilde{\theta})$$
$$= c\tilde{V}^{*-1}H^*\left(\sum_{j=1}^{K} w_j(\tilde{\theta})\Gamma^* w_j(\tilde{\theta}) + \sum_{j=1}^{K} w_j(\tilde{\theta})\Gamma^*_{j-l} w_l(\tilde{\theta})\right) H^*\tilde{V}^{*-1} + o_{p^*}(1)$$
$$= cH_0^*\tilde{V}^{-1}\left(\sum_{j=1}^{K} w_j(\tilde{\theta})\Gamma^* w_j(\tilde{\theta}) + \sum_{j=1}^{K} w_j(\tilde{\theta})\Gamma^*_{j-l} w_l(\tilde{\theta})\right)\tilde{V}^{-1}H_0^* + o_{p^*}(1).$$

We use Lemma B.1 in GP (2014) to obtain the final equality, $\tilde{V}^* = H^*\tilde{V}H^{*'} + O_{p^*}(\delta^{-2}_{NT_H}) = \tilde{V} + O_{p^*}(\delta^{-2}_{NT_H})$ and $H^* = H_0^* + O_{p^*}(\delta^{-2}_{NT_H})$ in probability.

Part (b):

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\sum_{j=1}^{K} w_j(\tilde{\theta})(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})\right]\left[\sum_{j=1}^{K} w_j(\tilde{\theta})H^*\tilde{f}_{t-j/m}\right]'$$
$$= \sum_{j=1}^{K} w_j(\tilde{\theta})\frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{f}_{t-j/m}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})' w_j(\tilde{\theta})$$
$$+ \sum_{j=1}^{K}\sum_{l\neq j}^{K} w_j(\tilde{\theta})\frac{1}{\sqrt{T}}\sum_{t=1}^{T} H^*\tilde{f}_{t-l/m}(\tilde{f}^*_{t-j/m} - H^*\tilde{f}_{t-j/m})' w_l(\tilde{\theta})$$
$$= cH^*\left[\sum_{j=1}^{K} w_j^2(\tilde{\theta}) + \sum_{j=1}^{K}\sum_{l\neq j}^{K} w_j(\tilde{\theta})\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-l/m}\tilde{f}'_{t-j/m}\right) w_l(\tilde{\theta})\right]\Gamma^*\left(\frac{1}{T_H}\sum_{s=1}^{T_H}\tilde{f}_s\tilde{f}^{*'}_s\right)\tilde{V}^{*-2} + o_{p^*}(1)$$
$$= cH_0^*\left[\sum_{j=1}^{K} w_j^2(\tilde{\theta}) + \sum_{j=1}^{K}\sum_{l\neq j}^{K} w_j(\tilde{\theta})\left(\frac{1}{T}\sum_{t=1}^{T}\tilde{f}_{t-l/m}\tilde{f}'_{t-j/m}\right) w_l(\tilde{\theta})\right]\Gamma^*\tilde{V}^{-2}H_0^* + o_{p^*}(1),$$

in probability. The final equality is by applying Lemma B.1. in GP (2014) and by $\frac{\tilde{f}^{*'}\tilde{f}}{T_H}\tilde{V}^{*-1} = \tilde{V}^{-1}H^*$ and $H^*\tilde{V}^{*-1} = \tilde{V}^{-1}H^*$. ■

In the remaining part, we prove Theorem 3.1. Recall that

$$e^*_{i,t_h} = \sum_{j=1}^{p_i} \tilde{\phi}_{i,j}(p_i) e^*_{i,t_h-j} + u^*_{i,t_h} \quad \text{for } t_h = 1, \ldots, T_H, \tag{8}$$

where $\tilde{\phi}_i(p_i) = (\tilde{\phi}_{i,j}(p_i), j = 1, \ldots, p_i)$ is Yule-Walker autoregressive parameter estimators. By the fact that $\tilde{\phi}_i(p_i)$ is Yule-Walker estimator, we can represent (8) as moving average process of order ∞ as

$$e^*_{i,t_h} = \sum_{j=0}^{\infty} \tilde{\psi}_{i,j}(p_i) u^*_{i,t_h-j}, \tag{9}$$

with $\tilde{\psi}_{i,0}(p_i) = 1$. By stacking (8) and (9) over $i = 1, \ldots, N$, we can rewrite it as vector representation as follows.

$$e^*_{t_h} = \sum_{j=1}^{p_i} \tilde{\Phi}_j(p) e^*_{t_h-j} + u^*_t, \text{ and} \tag{10}$$

$$e^*_{t_h} = \sum_{j=0}^{\infty} \tilde{\Psi}_j(p) u^*_{t_h-j}, \tag{11}$$

with $\tilde{\Psi}_0(p) = I_N$ and $p = \max(p_1, \ldots, p_N)$. Note that $\tilde{\Phi}_j(p)$ is $N \times N$ high-dimensional matrix, but it is a diagonal matrix by the construction such that $\tilde{\Phi}_j(p) = \text{diag}(\tilde{\phi}_{1,j}(p_1), \ldots, \tilde{\phi}_{N,j}(p_N))$. To prove Theorem 3.1, we include an auxiliary Lemma below.

Lemma D.5

(a) $\sum_{j=0}^{\infty} \|\tilde{\Psi}_j(p) - \Psi_j\| = o_p(1)$, *where* Ψ_j *is MA coefficients for* e_t *such that* $e_t = \sum_{j=0}^{\infty} \Psi_j u_{t-j}$.

(b) $\sum_{j=0}^{\infty} |\tilde{\psi}_{i,j}|^8 = O_p(1)$ *for* $i = 1, \ldots, N$.

Proof of Lemma D.5. To prove Lemma D.5-(a), we use the arguments in Bi, Shang, Yang, and Zhu (2021), specifically, Lemma C.7 in their supplement appendix. The difference is that their bootstrap method is applied to the factors, whereas our bootstrap method is constructed using the idiosyncratic error terms. Using their arguments in the proof of their Lemma C.7 and the fact that $\tilde{e}_{i,t} - e_{i,t} = \tilde{c}_{i,t} - c_{i,t} = O_p(\delta_{NT_H}^{-1})$, we can obtain the same result as in Lemma D.5, which yields $\sum_{j=0}^{\infty} \|\tilde{\Psi}_j(p) - \Psi_j\| = o_p(1)$. For (b), we can use Lemma D.5 and Assumption 3 in the main text to conclude. ■

Proof of Theorem 3.1. Following Lemma D.1, Remark 3 and 4, it is sufficient to show that our bootstrap algorithm described in Section 3 satisfy the bootstrap high level conditions C.1*-C.6*. **Condition C.1***. Part (a): We can show that $E^*(e^*_{i,t_h}) = \sum_{j=0}^{\infty} \tilde{\psi}_{i,j}(p_i) E^*(u^*_{i,t_h-j}) = 0$ since $E^*(u^*_{i,t_h-j}) = 0$ by its construction such that $u^*_{t_h} = \tilde{\Sigma}_u^{1/2} \eta_{t_h}$ with $\eta_{t_h} \sim$ i.i.d.$(0, I_N)$. Part (b): We first write γ^*_{st} as follows.

$$
\begin{aligned}
\gamma^*_{st} &= E^*\left(\frac{1}{N} e^{*\prime}_t e^*_s\right) \\
&= E^*\left[\frac{1}{N}\left(\sum_{j_1=0}^{\infty} \tilde{\Psi}_{j_1}(p) u^*_{t-j_1}\right)^{\prime}\left(\sum_{j_2=0}^{\infty} \tilde{\Psi}_{j_2}(p) u^*_{s-j_2}\right)\right] \\
&= E^*\left[\frac{1}{N} tr\left(\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \tilde{\Psi}_{j_1}(p) u^*_{t-j_1} u^{*\prime}_{s-j_2} \tilde{\Psi}^{\prime}_{j_2}\right)\right] \\
&= tr\left(\frac{1}{N}\sum_{j=0}^{\infty} \tilde{\Psi}_j(p)\tilde{\Sigma}_u \tilde{\Psi}_{s-t+j}(p)^{\prime}\right), \qquad (12)
\end{aligned}
$$

where we obtain the last equality since $E^*(u^*_{t-j_1} u^{*\prime}_{s-j_2}) = 0$ if $t - j_1 \neq s - j_2$. Using (12), we can write our condition as following:

$$
\begin{aligned}
\frac{1}{T_H}\sum_{s,t=1}^{T_H} |\gamma^*_{st}|^2 &= \frac{1}{T_H}\sum_{s,t=1}^{T_H}\left| tr\left(\frac{1}{N}\sum_{j=0}^{\infty} \tilde{\Psi}_j(p)\tilde{\Sigma}_u \tilde{\Psi}_{s-t+j}(p)^{\prime}\right)\right|^2 \\
&\leq \left(\frac{\|\tilde{\Sigma}_u\|^2}{N}\right)\left(\frac{1}{N}\frac{1}{T_H}\sum_{s,t=1}^{T_H}\left\|\sum_{j=0}^{\infty} \tilde{\Psi}_{s-t+j}(p)^{\prime}\tilde{\Psi}_j(p)\right\|^2\right) \\
&\leq \left(\frac{\|\tilde{\Sigma}_u\|^2}{N}\right)\frac{1}{N}\frac{1}{T_H}\sum_{s,t=1}^{T_H}\sum_{j=0}^{\infty}\left\|\tilde{\Psi}_{s-t+j}(p)\right\|^2\left\|\tilde{\Psi}_j(p)\right\|^2 = O_p(1).
\end{aligned}
$$

We can show that $\|\tilde{\Sigma}_u\|^2/N = O_p(1)$ since we can show the similar arguments in GP (2020) such that $\|\tilde{\Sigma}_u\| \leq \rho(\tilde{\Sigma}_u)\sqrt{\text{rank}(\tilde{\Sigma}_u)} \leq \rho(\tilde{\Sigma}_u)\sqrt{N}$ under Assumption 4-5 in the main text. We can also show that $\frac{1}{N}\frac{1}{T_H}\sum_{s,t=1}^{T_H}\sum_{j=0}^{\infty}\left\|\tilde{\Psi}_{s-t+j}(p)\right\|^2\left\|\tilde{\Psi}_j(p)\right\|^2 = O_p(1)$ under the summability condition. Part (c): First, note that we can write

$$\begin{aligned}
&E^*\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(e^*_{i,t_h}e^*_{i,s_h} - E^*(e^*_{i,t_h}e^*_{i,s_h}))\right|^2 \\
&= \frac{1}{N}\sum_{i,j=1}^{N} Cov^*(e^*_{i,t_h}e^*_{i,s_h}, e^*_{j,t_h}e^*_{j,s_h}) \\
&= \frac{1}{N}\sum_{i,j=1}^{N}\sum_{k_1,k_2,k_3,k_4=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_3}\tilde{\psi}_{j,k_4}Cov^*(u^*_{i,t_h-k_1}u^*_{i,s_h-k_2}, u^*_{j,t_h-k_3}u^*_{j,s_h-k_4}).
\end{aligned}$$

We can write $u_{i,t_h} = a_i'\eta_{t_h} = \sum_{l=1}^{N} a_{il}\eta_{l,t_h}$, where a_i' denotes the i-th row of $\tilde{\Sigma}_u^{1/2}$. For simpler notation, define $Cov^*(e^*_{i,t_h}e^*_{i,s_h}, e^*_{j,t_h}e^*_{j,s_h}) = \Delta_{ij,t_hs_h}$. We can rewrite Δ_{ij,t_hs_h} as follows.

$$\begin{aligned}
\Delta_{ij,t_hs_h} = &\sum_{k_1,k_2,k_3,k_4=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_3}\tilde{\psi}_{j,k_4}\sum_{l_1,l_2,l_3,l_4=1}^{N}a_{i,l_1}a_{i,l_2}a_{j,l_3}a_{j,l_4} \\
&\times \underbrace{Cov^*(\eta_{l_1,t_h-k_1}\eta_{l_2,s_h-k_2}, \eta_{l_3,t_h-k_3}\eta_{l_4,s_h-k_4})}_{A*}.
\end{aligned}$$

Since $\eta_{l,t_h} \sim$ i.i.d.$(0, I_N)$, we can consider $A*$ based on the choice of l_i for $i = 1,2,3,4$ and $t_h - k_1$, $s_h - k_2$, $t_h - k_3$ and $s_h - k_4$. We need $l_1 = l_2 = l_3 = l_4$, $l_1 = l_3 \neq l_2 = l_4$, or $l_1 = l_4 \neq l_2 = l_3$ for $A*$ to be non-zero. If $l_1 = l_2 = l_3 = l_4$, we need $t_h - k_1 = s_h - k_2 = t_h - k_3 = s_h - k_4$, $t_h - k_1 = t_h - k_3 \neq s_h - k_2 = s_h - k_4$, or $t_h - k_1 = s_h - k_4 \neq s_h - k_2 = t_h - k_3$. In this case, we have $A* = E^*(\eta^4_{l,t_h}) - 1$ or 1. If $l_1 = l_3 \neq l_2 = l_4$, we need $t_h - k_1 = t_h - k_3$ and $s_h - k_2 = s_h - k_4$, and we have $A* = 1$. Similarly, when $l_1 = l_4 \neq l_2 = l_3$, we need $t_h - k_1 = s_h - k_4$ and $s_h - k_2 = t_h - k_3$, and this yields $A* = 1$. Letting $\bar{\eta} \geq \max\{E^*(\eta^4_{l,t_h}) - 1, 1\}$, we can bound the condition as follows.

$$\frac{1}{T_H^2}\sum_{s_h,t_h=1}^{T_H}\frac{1}{N}\sum_{i,j=1}^{N}\Delta_{ij,t_hs_h}$$
$$\leq \bar{\eta}\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_1}\tilde{\psi}_{j,k_2}+\tilde{\psi}_{i,k_1}i,\tilde{k}_2\tilde{\psi}_{j,t-s+k_2}\tilde{\psi}_{j,s-t+k_1}\right)\left(\sum_{l=1}^{N}a_{i,l}a_{j,l}\right)^2$$
$$\leq \bar{\eta}\left[\underbrace{\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,k}\right)^2\left(\sum_{l=1}^{N}a_{i,l}a_{j,l}\right)^2}_{=A_{ij}-(I)}+\underbrace{\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,s-t+k}\right)\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,t-s+k}\right)\left(\sum_{l=1}^{N}a_{i,l}a_{j,l}\right)^2}_{=A_{ij}-(II)}\right].$$

Thus, the condition is bounded by

$$\bar{\eta}\left(\frac{1}{T^2}\sum_{t,s=1}^{T_H}\frac{1}{N}\sum_{i,j=1}^{N}A_{ij}-(I)+\frac{1}{T^2}\sum_{t,s=1}^{T_H}\frac{1}{N}\sum_{i,j=1}^{N}A_{ij}-(II)\right).$$

We can show that $\sum_{i,j=1}^{N}A_{ij}-(I)=O_p(1)$ which is sufficient to show that the first term is $O_p(1)$. Note that we can bound it further by Cauchy-Schwarz inequality as follows.

$$\frac{1}{N}\sum_{i,j=1}^{N}\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,k}\right)^2\left(\sum_{l=1}^{N}a_{i,l}a_{j,l}\right)^2\leq\left\{\frac{1}{N}\sum_{i,j=1}^{N}\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,k}\right)^4\right\}^{1/2}\left\{\frac{1}{N}\sum_{i,j=1}^{N}\left(\sum_{l=1}^{N}a_{i,l}a_{j,l}\right)^4\right\}^{1/2}.$$

We can show that for some positive constant M, by repetitive application of Hölder's inequality,

$$\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,k}\right)^4\leq M\left(\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}\tilde{\psi}_{j,k}|^4\right)\leq M\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}|^4|\tilde{\psi}_{j,k}|^4.$$

By Cauchy-Schwarz inequality, we can show that

$$\frac{1}{N}\sum_{i,j=1}^{N}\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,k}\right)^4\leq M\left(\frac{1}{N}\sum_{i=1}^{N}|\tilde{\psi}_{i,k}|^8\right)^{1/2}\left(\frac{1}{N}\sum_{j=1}^{N}|\tilde{\psi}_{j,k}|^8\right)^{1/2}.$$

We can show that this is $O_p(1)$ by Assumption 3 in the main text. We can also show that

$\left\{\frac{1}{N}\sum_{i,j=1}^{N}(a_i'a_j)^4\right\}^{1/2} = O_p(1)$, because we have

$$\left(\frac{1}{N}\sum_{i,j=1}^{N}(a_i'a_j)^2\right)^{1/2} \le \sqrt{tr(\tilde{\Sigma}_u^4)/N} \le \sqrt{\{tr(\tilde{\Sigma}_u^2)\}^2/N} = \|\tilde{\Sigma}_u\|/\sqrt{N} = O_p(1).$$

We can obtain the final equality by Assumption 5 and by applying the arguments in GP (2020) to $\tilde{\Sigma}_u$ such that $\|\tilde{\Sigma}_u\| \le \rho(\tilde{\Sigma}_u)\sqrt{\text{rank}(\tilde{\Sigma}_u)} \le \rho(\tilde{\Sigma}_u)\sqrt{N}$ (in their proof of Theorem 3.1). For the second term involved with $A_{ij}-(II)$, by applying Cauchy-Schwarz inequality, we have

$$\frac{1}{T_H^2}\sum_{s,t=1}^{T_H}\frac{1}{N}\sum_{i,j=1}^{N}A_{ij}-(II) \le \left\{\frac{1}{N}\sum_{i,j=1}^{N}\left(\sum_{l=1}^{N}a_{i,l}a_{j,l}\right)^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{i,j=1}^{N}\left(\frac{1}{T_H^2}\sum_{s,t=1}^{T_H}\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,s-t+k}\right)^2\right)^2\right\}^{1/2}.$$

We can show that $\left\{\frac{1}{N}\sum_{i,j=1}^{N}\left(\sum_{l=1}^{N}a_{i,l}a_{j,l}\right)^2\right\}^{1/2} = O_p(1)$ by using the similar arguments above. For the remaining term, we use Cauchy-Schwarz inequality as follows.

$$\frac{1}{N}\sum_{i,j=1}^{N}\left(\frac{1}{T_H^2}\sum_{s,t=1}^{T_H}\left(\sum_{k=0}^{\infty}\tilde{\psi}_{i,k}\tilde{\psi}_{j,s-t+k}\right)^2\right)^2 \le \frac{1}{N}\sum_{i,j=1}^{N}\frac{1}{T_H^2}\left(\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}|^2\frac{1}{T_H}\sum_{s,t=1}^{T_H}|\tilde{\psi}_{j,s-t+k}|^2\right)^2$$

Since we can show that $\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}|^2\frac{1}{T_H}\sum_{s,t=1}^{T_H}|\tilde{\psi}_{j,s-t+k}|^2 = O_p(1)$, the order of the above term is $O_p(N/T_H^2)$.

Condition C.2*. Part (a): By Cauchy-Schwarz inequality, we can bound the condition as follows.

$$\left\|\frac{1}{T_H}\sum_{t=1}^{T_H}\tilde{f}_s\tilde{f}_t'\gamma_{st}^*\right\| \le \left(\frac{1}{T_H}\sum_{s,t=1}^{T_H}\|\tilde{f}_s\tilde{f}_t'\|^2\right)^{1/2}\left(\frac{1}{T_H}\sum_{s,t=1}^{T_H}|\gamma_{st}^*|^2\right)^{1/2} = O_p(1).$$

We can show the term in the first parenthesis $O_p(1)$ since we can show that $\frac{1}{T_H}\sum_{t=1}^{T_H}\|\tilde{f}_t\|^4 = O_p(1)$ by using Lemma C.1-(i) in GP (2014) and use Cauchy-Schwarz inequality. The term in the second parenthesis is $O_p(1)$ by Condition C.1*-(b). Part (b): For simpler notation, in the remaining proof, we let $\tilde{\psi}_{i,j} = \tilde{\psi}_{i,j}(p_i)$ and $\tilde{\Psi}_j = \tilde{\Psi}_j(p)$. Note that we can rewrite the condition as follows.

$$\frac{1}{T_H}\sum_{t=1}^{T_H}\frac{1}{T_H}\sum_{s,l=1}^{T_H}\|\tilde{f}_s'\tilde{f}_l\|^2\frac{1}{N}\sum_{i,j=1}^{N}Cov^*(e_{i,t}^*e_{i,s}^*,e_{j,t}^*e_{j,l}^*).$$

By considering the combination of i, j and t, s and l, the covariance term $Cov^*(e_{i,t}^*e_{i,s}^*, e_{j,t}^*e_{j,l}^*)$ can be further bounded as follows.

$$\begin{aligned}Cov^*(e_{i,t}^*e_{i,s}^*,e_{j,t}^*e_{j,l}^*) \le \bar{\eta}&\left\{\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_1}\tilde{\psi}_{j,l-s+k_2}\right)\left(\sum_{m=1}^{N}a_{i,m}a_{j,m}\right)^2\right.\\ &\left.+\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,t-s+k_2}\tilde{\psi}_{j,l-t+k_1}\right)\left(\sum_{m=1}^{N}a_{i,m}a_{j,m}\right)^2\right\}\\ &=\bar{\eta}(B_{ij}-(I)+B_{ij}-(II)),\end{aligned}$$

where we denote $B_{ij}-(I)=\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_1}\tilde{\psi}_{j,l-s+k_2}\right)\left(\sum_{m=1}^{N}a_{i,m}a_{j,m}\right)^2$ and $B_{ij}-(II)=\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,t-s+k_2}\tilde{\psi}_{j,l-t+k_1}\right)\left(\sum_{m=1}^{N}a_{i,m}a_{j,m}\right)^2$. Then, using this bound on the covariance term, the condition is bounded by the following equation.

$$\bar{\eta}\left[\frac{1}{T_H^2}\sum_{t,s,l=1}^{T_H}\|\tilde{f}_s'\tilde{f}_l\|^2\frac{1}{N}\sum_{i,j=1}^{N}B_{ij}-(I)+\frac{1}{T_H^2}\sum_{t,s,l=1}^{T_H}\|\tilde{f}_s'\tilde{f}_l\|^2\frac{1}{N}\sum_{i,j=1}^{N}B_{ij}-(II)\right]$$

The first term in the square bracket can be bounded by Cauchy-Schwarz inequality as follows.

$$\frac{1}{T_H^2}\sum_{t,s,l=1}^{T_H}\|\tilde{f}_s'\tilde{f}_l\|^2\frac{1}{N}\sum_{i,j=1}^{N}B_{ij}-(I)\le\frac{1}{T_H}\sum_{t=1}^{T_H}\left(\frac{1}{T_H}\sum_{s,l=1}^{T_H}\|\tilde{f}_s'\tilde{f}_l\|^4\right)^{1/2}\left(\frac{1}{T_H}\sum_{s,l=1}^{T_H}\left|\frac{1}{N}\sum_{i,j=1}^{N}B_{ij}-(I)\right|^2\right)^{1/2}$$

We can show that $\frac{1}{T_H}\sum_{s,l=}^{T_H}\|\tilde{f}_s'\tilde{f}_l\|^4=O_p(1)$ by applying Lemma C.1 in GP (2014) with $p=8$ (this can be verified under our Assumption 1 in the main text). To show that $\frac{1}{T_H}\sum_{s,l=1}^{T_H}\left|\frac{1}{N}\sum_{i,j=1}^{N}B_{ij}-(I)\right|^2=O_p(1)$, we first bound it by Cauchy-Schwarz inequality as follows.

$$\frac{1}{T_H}\sum_{s,l=1}^{T_H}\left|\frac{1}{N}\sum_{i,j=1}^{N}B_{ij}-(I)\right|^2 \leq \left(\frac{1}{N}\sum_{i,j=1}^{N}(a_i'a_j)^2\right)\left(\frac{1}{N}\sum_{i,j=1}^{N}\frac{1}{T_H}\sum_{s,l=1}^{T_H}\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_1}\tilde{\psi}_{j,l-s+k_2}\right)^2\right).$$

As in the proof of Condition C.1*-(c), we can show that $\frac{1}{N}\sum_{i,j=1}^{N}(a_i'a_j)^2 = O_p(1)$. First, note that by using Hölder's inequality, we can show that $\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_1}\tilde{\psi}_{j,l-s+k_2}\right)^2 \leq M$ $\sum_{k_1,k_2=0}^{\infty}|\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_1}\tilde{\psi}_{j,l-s+k_2}|^2$, for some positive constant M. Then, we apply Cauchy-Schwarz inequality and Hölder's inequality to obtain the following inequality. For some positive constant M,

$$\begin{aligned}
&\frac{1}{N}\sum_{i,j=1}^{N}\frac{1}{T_H}\sum_{s,l=1}^{T_H}\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,k_1}\tilde{\psi}_{j,l-s+k_2}\right)^2 \\
&\quad\leq M\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{k_1=0}^{\infty}|\tilde{\psi}_{i,k_1}|^4\sum_{k_2=0}^{\infty}|\tilde{\psi}_{i,k_2}|^4\right)^{1/2}\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{k_1=0}^{\infty}|\tilde{\psi}_{j,k_1}|^4\left(\sum_{k_2=0}^{\infty}\frac{1}{T_H}\sum_{s,l=1}^{T_H}|\tilde{\psi}_{j,l-s+k_2}|^2\right)^2\right)^{1/2}.
\end{aligned}$$

Note that $\frac{1}{T_H}\sum_{s,l=1}^{T_H}|\tilde{\psi}_{j,l-s+k_2}|^2 = \sum_{\tau=0}^{T_H-1}\left(1-\frac{\tau}{T_H}\right)|\tilde{\psi}_{j,\tau+k_2}|^2 \leq \sum_{\tau=0}^{\infty}|\tilde{\psi}_{j,\tau+k_2}|^2$. Then, since $\sum_{k_2=0}^{\infty}\sum_{\tau=0}^{\infty}|\tilde{\psi}_{j,\tau+k_2}|^2 = \sum_{k_3=0}^{\infty}(k_3+1)|\tilde{\psi}_{j,k_3}|^2$, we can show that $\left(\sum_{k_2=0}^{\infty}\frac{1}{T_H}\sum_{s,l=1}^{T_H}|\tilde{\psi}_{j,l-s+k_2}|^2\right)^2 \leq$ $M_1\sum_{k_3=0}^{\infty}(k_3+1)^2|\tilde{\psi}_{j,k_3}|^4$ for some positive constant M_1. Therefore, we can show that the second term is $O_p(1)$ by Assumption 3 with $r=2$. By Assumption 3, we can show that $\frac{1}{N}\sum_{i=1}^{N}\sum_{k_1=0}^{\infty}|\tilde{\psi}_{i,k_1}|^4\sum_{k_2=0}^{\infty}|\tilde{\psi}_{i,k_2}|^4 = O_p(1)$ and we can also show that the remaining term in the above inequality is $O_p(1)$. Next, we show that $\frac{1}{T_H}\sum_{s,l=1}^{T_H}\left|\frac{1}{N}\sum_{i,j=1}^{N}B_{ij}-(I)\right|^2 = O_p(1)$. By applying Cauchy-Schwarz inequality repetitively, it is sufficient to show that

$$\begin{aligned}
&\frac{1}{N}\sum_{i,j=1}^{N}\frac{1}{T_H}\sum_{s,l=1}^{T_H}\left|\frac{1}{T_H}\sum_{t=1}^{T_H}\left(\sum_{k_1,k_2=0}^{\infty}\tilde{\psi}_{i,k_1}\tilde{\psi}_{i,k_2}\tilde{\psi}_{j,t-s+k_2}\tilde{\psi}_{j,l-t+k_1}\right)\right|^2 \\
&\quad\leq M\frac{1}{T_H}\left(\frac{1}{N}\sum_{i,j=1}^{N}\left|\frac{1}{T_H}\sum_{t,l=1}^{T_H}\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}\tilde{\psi}_{j,l-t+k}|^2\right|^2\right)^{1/2}\left(\frac{1}{N}\sum_{i,j=1}^{N}\left|\frac{1}{T_H}\sum_{t,s=1}^{T_H}\sum_{k_2=0}^{\infty}|\tilde{\psi}_{i,k}\tilde{\psi}_{j,t-s+k}|^2\right|^2\right)^{1/2} \\
&= O_p(1).
\end{aligned}$$

In fact, we can use Cauchy-Schwarz inequality and Assumption 3 to show that the term is $O_p(1)$. For example, we can show that

$$\begin{aligned}\frac{1}{N}\sum_{i,j=1}^{N}\left|\frac{1}{T_H}\sum_{t,l=1}^{T_H}\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}\tilde{\psi}_{j,l-t+k}|^2\right|^2 &= \frac{1}{N}\sum_{i,j=1}^{N}\left|\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}|^2\frac{1}{T_H}\sum_{t,l=1}^{T_H}|\tilde{\psi}_{j,l-t+k}|^2\right|^2 \\ &\leq \left(M_1\frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{\infty}|\tilde{\psi}_{i,k}|^8\right)^{1/2}\left(M_2\frac{1}{N}\sum_{j=1}^{N}\sum_{k=0}^{\infty}|(1+k)^4|\tilde{\psi}_{j,k}|^8\right)^{1/2} \\ &= O_p(1),\end{aligned}$$

for some positive constants M_1 and M_2. We obtain the final equality by Assumption 3 with $r=4$. Part (c): First, note that we can write the condition as follows.

$$E^*\left\|\frac{1}{\sqrt{T_H N}}\sum_{t=1}^{T_H}\sum_{i=1}^{N}\tilde{f}_t\tilde{\lambda}_i' e_{i,t}^*\right\|^2 = \frac{1}{T_H}\sum_{t,s=1}^{T_H}tr(\tilde{f}_t\tilde{f}_s')E^*\left(\frac{e_s^{*\prime}\tilde{\Lambda}}{\sqrt{N}}\frac{\tilde{\Lambda}' e_t^*}{\sqrt{N}}\right)$$

Since $E^*\left(\frac{e_s^{*\prime}\tilde{\Lambda}}{\sqrt{N}}\frac{\tilde{\Lambda}' e_t^*}{\sqrt{N}}\right) = E^*\left[tr\left(\frac{\tilde{\Lambda}' e_t^*}{\sqrt{N}}\frac{e_s^{*\prime}\tilde{\Lambda}}{\sqrt{N}}\right)\right]$, we focus on $E^*(e_t^* e_s^{*\prime})$. Under vector MA(∞) representation of e_t^*, we can write it as follows.

$$E^*(e_t^* e_s^{*\prime}) = \sum_{k_1,k_2=0}^{\infty}\tilde{\Psi}_{k_1}E^*(u_{t-k_1}^* u_{s-k_2}^{*\prime})\tilde{\Psi}_{k_2} = \sum_{k=0}^{\infty}\tilde{\Psi}_k\tilde{\Sigma}_u\tilde{\Psi}_{s-t+k}'$$

By plugging this back into the condition and using Cauchy-Schwarz inequality,

$$\begin{aligned}E^*\left\|\frac{1}{\sqrt{T_H N}}\sum_{t=1}^{T_H}\sum_{i=1}^{N}\tilde{f}_t\tilde{\lambda}_i' e_{i,t}^*\right\|^2 &= \frac{1}{T_H}\sum_{t,s=1}^{T_H}tr(\tilde{f}_t\tilde{f}_s')tr\left(\frac{\tilde{\Lambda}'\sum_{k=0}^{\infty}\tilde{\Psi}_k\tilde{\Sigma}_u\tilde{\Psi}_{s-t+k}'\tilde{\Lambda}}{N}\right) \\ &\leq \left(\frac{1}{T_H}\sum_{t,s=1}^{T_H}|tr(\tilde{f}_t\tilde{f}_s')|^2\right)^{1/2}\left(\frac{1}{T_H}\sum_{t,s=1}^{T_H}\left|tr(\tilde{\Gamma}_{s-t})\right|^2\right)^{1/2},\end{aligned}$$

where we denote $\tilde{\Gamma}_{s-t} = \frac{1}{N}\tilde{\Lambda}'\left(\sum_{k=0}^{\infty}\tilde{\Psi}_k\tilde{\Sigma}_u\tilde{\Psi}_{s-t+k}'\right)\tilde{\Lambda}$. We can show that the first term is $O_p(1)$ by Assumption 1 and using the results in Lemma C.1 in GP (2014). For the second term, it is sufficient to show that $tr(\tilde{\Gamma}_t) = O_p(1)$. This is implied by Condition C.6*-(b), which will be verified. Part (d): We can rewrite the condition as follows.

$$\frac{1}{T_H}\sum_{t=1}^{T_H} E^* \left\| \frac{\tilde{\Lambda}' e_t^*}{\sqrt{N}} \right\|^2 = \frac{1}{T_H}\sum_{t=1}^{T_H} E^* \left[tr\left(\frac{\tilde{\Lambda}' e_t^* e_t^{*\prime} \tilde{\Lambda}}{N} \right) \right] = \frac{1}{T_H}\sum_{t=1}^{T_H} tr\left(\frac{\tilde{\Lambda}' E^*(e_t^* e_t^{*\prime}) \tilde{\Lambda}}{N} \right)$$

As we have shown previously in the proof of Condition C.2*-(c), we can write $E^*(e_t^* e_t^{*\prime}) = \sum_{k=0}^{\infty} \tilde{\Psi}_k \tilde{\Sigma}_u \tilde{\Psi}_k'$. Therefore, the condition is $\frac{1}{T_H}\sum_{t=1}^{T_H} tr\left(\frac{\tilde{\Lambda}' \sum_{k=0}^{\infty} \tilde{\Psi}_k \tilde{\Sigma}_u \tilde{\Psi}_k' \tilde{\Lambda}}{N} \right)$, and this is $O_p(1)$ given that $tr(\tilde{\Gamma}_0) = O_p(1)$. Part (e): To verify this condition, we use $r = 1$ (recall that r is the number of factors) for a simpler notation. Therefore, it suffices to show that $Var^*(A^*) = o_p(1)$, where $A^* = \frac{1}{T_H}\sum_{t=1}^{T_H} \left(\frac{\tilde{\Lambda}' e_t^*}{\sqrt{N}} \right) \left(\frac{e_t^{*\prime} \tilde{\Lambda}}{\sqrt{N}} \right)$. Note that

$$\begin{aligned}
Var^*(A^*) &= \frac{1}{T_H^2}\sum_{t,s=1}^{T_H} \frac{1}{N^2}\sum_{i,j,k,l}^{N} \tilde{\lambda}_i \tilde{\lambda}_j \tilde{\lambda}_k \tilde{\lambda}_l Cov^*(e_{i,t}^* e_{j,t}^*, e_{l,s}^* e_{k,s}^*) \\
&\leq 2\bar{\eta} \frac{1}{T_H^2}\sum_{t,s=1}^{T_H} \frac{1}{N^2}\sum_{i,j,k,l}^{N} \tilde{\lambda}_i \tilde{\lambda}_j \tilde{\lambda}_k \tilde{\lambda}_l \left(\sum_{p_1,p_2=0}^{\infty} \tilde{\psi}_{i,p_1} \tilde{\psi}_{j,p_2} \tilde{\psi}_{l,s-t+p_1} \tilde{\psi}_{k,s-t+p_2} \right) \\
&\quad \times \left(\sum_{m_1,m_2=1}^{N} a_{i,m_1} a_{j,m_2} a_{l,m_1} a_{k,m_2} \right) \\
&= 2\bar{\eta} \frac{1}{T_H^2}\sum_{t,s=1}^{T_H} \left\{ \left(\frac{1}{N}\sum_{i,l=1}^{N} \tilde{\lambda}_i \tilde{\lambda}_l \right) \left(\sum_{p_1=0}^{\infty} \tilde{\psi}_{i,p_1} \tilde{\psi}_{l,s-t+p_1} \right) \left(\sum_{m_1=1}^{N} a_{i,m_1} a_{l,m_1} \right) \right\}^2 \\
&= 2\bar{\eta} \frac{1}{T_H^2}\sum_{t,s=1}^{T_H} \left(\frac{\tilde{\Lambda}' \sum_{p=0}^{\infty} \tilde{\Psi}_p \tilde{\Sigma}_u \tilde{\Psi}_{s-t+p} \tilde{\Lambda}}{N} \right)^2,
\end{aligned}$$

where we obtain the second inequality by taking account of the covariance term given the combination of i, j, k, and l and t and s, similar to the proof of Condition C.1*-(c). Note that given that $tr(\tilde{\Gamma}_{s-t}) = O_p(1)$, we can show that $\frac{1}{T_H}\sum_{t,s=1}^{T_H} \tilde{\Gamma}_{s-t}^2 = O_p(1)$. Therefore, $Var^*(A^*) = O_p(1/T_H) = o_p(1)$. The proof to verify **Condition C.3*** is very similar to the proof of Condition C.2*. For example, Condition C.3*-(b) and (c) can be verified given that $tr(\tilde{\Gamma}_\tau) = O_p(1)$ with $\tau \neq 0$.

Condition C.4* Part (a): Given that ε_t^* and $e_{t-j/m}^*$ are independent in Assumption 2, it is sufficient to show that

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{NT}\sum_{s,l=1}^{T}\sum_{i=1}^{N}Cov^*(e^*_{i,t-j/m}e^*_{i,s-j/m}, e^*_{i,t-j/m}e^*_{i,l-j/m}) = O_p(1).$$

We show a similar term is $O_p(1)$ in Condition C.2*-(b). Part (b): Similarly, given the independence of ε^*_t and $e^*_{i,t-j/m}$, it suffices to show that $E^*\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\tilde{\Lambda}' e^*_{t-j/m}}{\sqrt{N}}\right\|^2 = O_p(1)$, which is verified in Condition C.2*-(c). **Condition C.5*** and **Condition C.6***-(a) can be verified using the arguments in GP (2014), since ε^*_t is constructed in the same way. **Condition C.6***- Part (b): Note that $\tilde{\Gamma}_k$ can be rewritten as follows.

$$\tilde{\Gamma}_k = \frac{1}{T_H}\sum_{t=1}^{T_H}\frac{1}{N}\tilde{\Lambda}'\sum_{p=0}^{\infty}\tilde{\Psi}_p\tilde{\Sigma}_u\tilde{\Psi}'_{p-k}\tilde{\Lambda} = \frac{\tilde{\Lambda}'\tilde{\Sigma}_{e,k}\tilde{\Lambda}}{N},$$

where we let $\tilde{\Sigma}_{e,k} \equiv \sum_{p=0}^{\infty}\tilde{\Psi}_p\tilde{\Sigma}_u\tilde{\Psi}'_{p-k}$. Let $\bar{\Gamma}_k = \frac{\Lambda'\tilde{\Sigma}_{e,k}\Lambda}{N}$. Then, by adding and subtracting appropriately, we have the following:

$$\begin{aligned}\tilde{\Gamma}_k - H_0\Gamma_k H_0' &= \tilde{\Gamma}_k - H_0\bar{\Gamma}_k H_0' + H_0\bar{\Gamma}_k H_0' - H_0\Gamma_k H_0' \\ &= \underbrace{(\tilde{\Gamma}_k - H_0\bar{\Gamma}_k H_0')}_{\equiv D_1} + \underbrace{H_0(\bar{\Gamma}_k - \Gamma_k)H_0'}_{\equiv D_2}.\end{aligned}$$

We can show that D_1 and D_2 are $o_p(1)$. In order to show that $D_2 = o_p(1)$, it is sufficient to show that $\tilde{\Sigma}_{e,k} - \Sigma_{e,k} \to 0$, where $\Sigma_{e,k} \equiv \sum_{p=0}^{\infty}\Psi_p\Sigma_u\Psi'_{p-k}$ with $\Sigma_u = E(u_t u_t')$. Note that we can expand $\tilde{\Sigma}_{e,k} - \Sigma_{e,k}$ as follows.

$$\tilde{\Sigma}_{e,k} - \Sigma_{e,k} = \underbrace{\sum_{p=0}^{\infty}(\tilde{\Psi}_p - \Psi_p)\tilde{\Sigma}_u\tilde{\Psi}_{p-k}}_{D_{21}} + \underbrace{\sum_{p=0}^{\infty}\Psi_p(\tilde{\Sigma}_u - \Sigma_u)\tilde{\Psi}_{p-k}}_{D_{22}} + \underbrace{\sum_{p=0}^{\infty}\Psi_p\Sigma_u(\tilde{\Psi}_{p-k} - \Psi_{p-k})}_{D_{23}}.$$

We can show that $D_{22} = o_p(1)$ since $\rho(\tilde{\Sigma}_u - \Sigma_u) \xrightarrow{p} 0$ under Assumptions 4-5 using the arguments in GP (2020). We can show that D_{21} and D_{23} are of order $o_p(1)$ by Lemma D.5.

Next, we show that D_1 is $o_p(1)$. We can decompose D_1 further as follows.

$$D_1 = \underbrace{\frac{1}{N}(\tilde{\Lambda} - \Lambda H^{-1})'\tilde{\Sigma}_{e,k}(\tilde{\Lambda} - \Lambda H^{-1})}_{D_{11}} + \underbrace{\frac{1}{N}H^{-1\prime}\Lambda'\tilde{\Sigma}_{e,k}(\tilde{\Lambda} - \Lambda H^{-1})}_{D_{12}} + \underbrace{\frac{1}{N}(\tilde{\Lambda} - \Lambda H^{-1})'\tilde{\Sigma}_{e,k}\Lambda H^{-1}}_{D_{12}'}.$$

$D_{11} = o_p(1)$ by applying Cauchy-Schwarz inequality as follows.

$$\|D_{11}\| \leq \underbrace{\left\|\frac{1}{\sqrt{N}}(\tilde{\Lambda} - \Lambda H^{-1})\right\|^2}_{=o_p(1)} \underbrace{\left\|\sum_{p=0}^{\infty} \tilde{\Psi}_p \tilde{\Sigma}_u \tilde{\Psi}'_{p-k}\right\|}_{=O_p(1)} = o_p(1),$$

where we use the fact that

$$\left\|\sum_{p=0}^{\infty} \tilde{\Psi}_p \tilde{\Sigma}_u \tilde{\Psi}'_{p-k}\right\| \leq \left\|\sum_{p=0}^{\infty} \tilde{\Psi}_p \tilde{\Psi}'_{p-k}\right\| \left\|\tilde{\Sigma}_u\right\| \leq \sum_{p=0}^{\infty} \left\|\tilde{\Psi}_p \tilde{\Psi}'_{p-k}\right\| \rho(\tilde{\Sigma}_u) = O_p(1),$$

and use the arguments in GP (2020). Since we have

$$\|D_{12}\| \leq \|H^{-1}\| \left\|\Lambda/\sqrt{N}\right\| \left\|\tilde{\Sigma}_{e,k}\right\| \left(\left\|\frac{1}{\sqrt{N}}(\tilde{\Lambda} - \Lambda H^{-1})\right\|^2\right)^{1/2},$$

we can show that this is $o_p(1)$ using similar arguments as we did for D_{11}. ■

E Additional simulation results

E.1 Simulation: results of DGP 1 and 2 of the factor-MIDAS regression model

Table 1 presents the results of DGP 1 and 2 in each panel. The results indicate that there is no bias when using the true factor, however, a bias does exist when using the estimated factor as a regressor. Increasing the sample size in both cross-sectional and time series dimensions results in a decrease in bias. If the cross-sectional dimension is small (50 and 100), the plug-in bias tends to overestimate the bias size. Both bootstrap methods perform similarly and replicate bias size well. When no method is used to correct the bias, size distortion occurs in terms of coverage rates. The plug-in bias somewhat recovers the size distortion, but bootstrap methods outperform the plug-in bias method. The results of DGP 1 and DGP 2 are similar, and both bootstrap methods are valid for these scenarios since the idiosyncratic error terms are randomly generated from a standard normal distribution.

E.2 Simulation experiment: increase in autoregressive coefficient

Table 2 shows the bias and 95% coverage rate of β when the idiosyncratic error term follows simple AR (1) process as:

$$e_{i,t_h} = \rho_i e_{i,t_h-1} + v_{i,t_h} \text{ for } t_h = 1, \ldots, T_H$$

where v_{i,t_h} is i.i.d. randomly generated from $N(0,1)$. We let ρ_i indicate the auto-regressive coefficient, which implies the persistence of auto-regressive process. For simplicity, we impose that each variable shares same autoregressive coefficient, $\rho_i = \rho$. In Table 2, we compare the results by varying persistence. We increase the coefficient from 0 to 0.7. When the persistence in the idiosyncratic error term is $\rho = 0.5$, the bias is around twice bigger than the bias where there is no serial-dependence. Moreover, the size of bias increase as the persistence increases.

E.3 Simulation experiment: unrestricted MIDAS regression model

Table 3-5 show the performance of bootstrap methods (wild bootstrap and AR-sieve + CSD bootstrap method) as well as plug-in bias estimation method under the framework of unrestricted MIDAS regression model. We consider the unrestricted MIDAS regression with

Table 1: DGP 1 & DGP 2 - Bias and coverage rate of 95% CIs for β

		$N=50$			$N=100$			$N=200$		
		$T=50$	100	200	50	100	200	50	100	200
		$T_H=150$	300	600	150	300	600	150	300	600
	bias									
DGP 1: homo & homo	True Factor	-0.01	-0.01	0.00	-0.02	-0.01	0.00	0.00	0.00	0.00
	Estimated Factor	-0.32	-0.31	-0.29	-0.20	-0.17	-0.16	-0.12	-0.10	-0.08
	Plug-in	-0.38	-0.34	-0.32	-0.21	-0.19	-0.18	-0.10	-0.10	-0.09
	WB	-0.25	-0.24	-0.23	-0.16	-0.15	-0.14	-0.11	-0.09	-0.08
	AR-sieve+CSD	-0.24	-0.24	-0.23	-0.16	-0.15	-0.14	-0.10	-0.09	-0.08
	95% coverage rate									
	Estimated Factor	84.8	82.0	73.9	89.6	90.5	88.3	91.7	92.7	93.4
	Plug-in	87.6	89.1	89.3	90.4	92.1	92.4	91.2	92.7	93.6
	WB	94.1	94.7	93.3	95.0	95.6	94.5	92.7	95.4	94.9
	AR-sieve+CSD	95.8	94.9	92.4	95.8	96.1	95.0	96.0	96.3	95.3
	bias									
DGP 2: hetero & homo	True Factor	-0.01	0.00	0.00	0.00	0.01	-0.01	0.01	-0.01	0.00
	Estimated Factor	-0.34	-0.30	-0.29	-0.19	-0.16	-0.16	-0.10	-0.10	-0.09
	Plug-in	-0.37	-0.34	-0.32	-0.20	-0.19	-0.18	-0.10	-0.10	-0.09
	WB	-0.24	-0.24	-0.23	-0.16	-0.15	-0.14	-0.10	-0.09	-0.08
	AR-sieve+CSD	-0.24	-0.24	-0.23	-0.16	-0.15	-0.14	-0.10	-0.09	-0.08
	95% coverage rate									
	Estimated Factor	78.1	76.2	68.4	85.9	88.1	86.2	88.7	91.5	91.6
	Plug-in	82.7	86.8	88.3	86.6	89.8	92.5	88.9	92.3	92.5
	WB	91.7	93.0	93.1	92.6	93.3	94.2	91.0	94.4	94.0
	AR-sieve+CSD	92.5	92.9	92.2	94.0	95.2	93.8	93.5	94.8	94.8

In DGP 1, both error terms are homoskedastic. In DGP 2, MIDAS regression error terms are heteroskedastic and idiosyncratic error terms are homoskedastic. The results of coverage rates, when we use the estimated factors and plug-in bias, are based on asymptotic theory. The bootstrap coverage rates use the bootstrap equal-tailed percentile t method.

a single factor as follows.

$$y_t = \beta + \sum_{j=1}^{K} \alpha_k f_{t-j/3} + \varepsilon_t$$

$$X_{t-k/3} = \Lambda f_{t-k/3} + e_{t-k/3},$$

Table 2: Bias and 95% coverage rate of β

N	T_H	$\rho=0$		$\rho=0.5$		$\rho=0.6$		$\rho=0.7$	
		bias	95%	bias	95%	bias	95%	bias	95%
50	150	-0.3380	84.7	-0.5887	68.02	-0.6808	60.42	-0.7993	49.18
	300	-0.3100	81.76	-0.5362	57.94	-0.6197	48.16	-0.7278	35.18
	600	-0.2890	74	-0.4970	40.96	-0.5746	29.32	-0.6761	17.2
100	150	-0.2022	89.82	-0.3763	83.18	-0.4450	79.34	-0.5372	72.62
	300	-0.1709	90.72	-0.3157	81.1	-0.3729	75.68	-0.4502	67.1
	600	-0.1565	88.7	-0.2849	75.36	-0.3358	67.44	-0.4047	56.16
200	150	-0.1343	91.48	-0.2639	87.6	-0.3163	85.38	-0.3890	81.8
	300	-0.1027	92.5	-0.1996	89.18	-0.2393	87.28	-0.2943	83.54
	600	-0.0865	92.44	-0.1647	88.02	-0.1968	85.48	-0.2411	80.7

for $k=0,1,2$ and $t=1,\dots,T$. The simulation design is identical to that in Section 4 in the main text: $f_{t-k/m}\sim$ i.i.d. $N(0,1)$ and $\lambda_i\sim$ i.i.d. $U[0,1]$. We consider six data generating processes as detailed in Table 1 in the main text. In this setup, y_t is predicted using six lags of the factor ($K=6$). We set $\beta=0$ and $\alpha_k=\alpha^k$ with $\alpha=0.8$. The estimation procedure is similar to restricted MIDAS, which proceeds in two steps: we first estimate the factors from $X_{t-k/m}$ and then in the second step, we regress y_t on the temporally aggregated estimated factors up to six lags. We report the bias in α_1 associated with the true factor, estimated factor, plug-in estimation method, as well as two bootstrap methods: the wild bootstrap method and the AR-sieve + CSD bootstrap method. In addition, we provide the 95% coverage rates associated with the estimated factor, plug-in estimation method, and both bootstrap methods. Note that the wild bootstrap is not valid in DGPs 4 to 6.

DGPs 1 to 3 yield comparable outcomes: the plug-in estimation method and the two bootstrap methods are perform similarly, and effectively capture the size of the bias. Regarding the coverage rate, the bootstrap methods outperform the plug-in estimation method.

In DGP 4, where the idiosyncratic error terms of the factor model are serially dependent, the AR-sieve + CSD bootstrap method outperforms the plug-in estimation method in terms of replicating the bias and correcting the distortion induced by the bias. In DGP 5, the plug-in estimation method performs the best in estimating the bias size. In terms of coverage rate, the plug-in estimation method outperforms the wild bootstrap method when N is small, while the AR-sieve + CSD bootstrap method outperforms other two methods across all sample sizes. Finally, in DGP 6, both the plug-in and the AR-sieve + CSD bootstrap methods replicate the bias size well, with the AR-sieve + CSD bootstrap method performing the best at recovering the distortion in the coverage rate.

Table 3: DGP 1 & DGP 2 - Bias and coverage rate of 95% CIs for β

		$N = 50$			$N = 100$			$N = 200$		
		$T = 50$	100	200	50	100	200	50	100	200
		$T_H = 150$	300	600	150	300	600	150	300	600
	bias									
DGP 1: homo & homo	True Factor	0.00	-0.00	-0.00	-0.00	0.00	-0.00	0.00	0.00	0.00
	Estimated Factor	-0.10	-0.10	-0.09	-0.07	-0.05	-0.05	-0.04	-0.03	-0.03
	Plug-in	-0.09	-0.08	-0.08	-0.05	-0.05	-0.04	-0.03	-0.02	-0.02
	WB	-0.08	-0.08	-0.07	-0.05	-0.05	-0.04	-0.04	-0.03	-0.03
	AR-sieve+CSD	-0.08	-0.08	-0.07	-0.05	-0.05	-0.04	-0.04	-0.03	-0.03
	95% coverage rate									
	Estimated Factor	86.8	81.5	71.0	91.6	90.6	88.1	93.3	94.1	93.6
	Plug-in	89.7	89.7	90.0	92.1	92.6	92.5	93.2	94.0	94.4
	WB	94.3	93.5	92.6	95.3	94.5	93.8	95.7	95.3	95.1
	AR-sieve+CSD	94.4	93.1	92.6	95.6	94.4	94.0	95.6	95.3	95.1
	bias									
DGP 2: hetero & homo	True Factor	-0.00	-0.00	-0.00	-0.00	0.00	-0.00	0.00	-0.00	0.00
	Estimated Factor	-0.11	-0.10	-0.09	-0.07	-0.05	-0.05	-0.04	-0.03	-0.03
	Plug-in	-0.09	-0.08	-0.08	-0.05	-0.05	-0.04	-0.03	-0.02	-0.02
	WB	-0.08	-0.08	-0.07	-0.05	-0.05	-0.04	-0.03	-0.03	-0.03
	AR-sieve+CSD	-0.08	-0.08	-0.07	-0.05	-0.05	-0.04	-0.03	-0.03	-0.03
	95% coverage rate									
	Estimated Factor	78.0	74.5	65.4	86.6	87.3	86.2	89.2	91.6	92.0
	Plug-in	84.3	86.8	89.3	88.2	90.2	91.5	89.7	92.1	93.7
	WB	90.7	91.2	91.7	92.5	92.7	92.8	92.6	93.9	94.6
	AR-sieve+CSD	90.9	91.3	91.5	92.8	92.7	93.1	92.6	93.7	94.5

In DGP 1, both error terms are homoskedastic. In DGP 2, MIDAS regression error terms are heteroskedastic and idiosyncratic error terms are homoskedastic. The results of coverage rates, when we use the estimated factors and plug-in bias, are based on asymptotic theory. The bootstrap coverage rates use the bootstrap equal-tailed percentile t method.

F Other empirical result

In Table 6, we present the results after excluding the COVID pandemic period. The results are similar to Table 5 in the main text. When using the bootstrap method, the confidence

Table 4: DGP 3 & DGP 4 - Bias and coverage rate of 95% CIs for β

		$N=50$			$N=100$			$N=200$		
		$T=50$	100	200	50	100	200	50	100	200
		$T_H=150$	300	600	150	300	600	150	300	600
	bias									
DGP 3: hetero & hetero	True Factor	-0.00	0.00	-0.00	-0.00	0.00	0.00	0.00	0.00	0.00
	Estimated Factor	-0.11	-0.11	-0.10	-0.07	-0.05	-0.05	-0.04	-0.03	-0.03
	Plug-in	-0.10	-0.09	-0.09	-0.05	-0.05	-0.05	-0.03	-0.03	-0.03
	WB	-0.09	-0.09	-0.08	-0.06	-0.05	-0.05	-0.04	-0.03	-0.03
	AR-sieve+CSD	-0.09	-0.08	-0.08	-0.06	-0.05	-0.05	-0.04	-0.03	-0.03
	95% coverage rate									
	Estimated Factor	75.9	72.7	61.4	85.3	87.6	84.4	89.2	91.6	91.3
	Plug-in	84.6	87.6	88.5	87.7	90.5	91.8	89.3	92.2	93.4
	WB	91.1	92.0	91.7	91.8	92.7	93.3	92.9	93.6	94.2
	AR-sieve+CSD	91.1	91.7	90.6	91.9	92.6	93.1	92.7	93.9	94.1
	bias									
DGP 4: hetero & AR	True Factor	-0.00	0.00	-0.00	-0.00	0.00	0.00	0.00	0.00	0.00
	Estimated Factor	-0.15	-0.14	-0.13	-0.10	-0.07	-0.07	-0.06	-0.05	-0.04
	Plug-in	-0.08	-0.08	-0.08	-0.05	-0.05	-0.05	-0.03	-0.03	-0.02
	WB	-0.08	-0.08	-0.08	-0.05	-0.05	-0.05	-0.04	-0.03	-0.03
	AR-sieve+CSD	-0.10	-0.10	-0.09	-0.07	-0.07	-0.06	-0.05	-0.04	-0.04
	95% coverage rate									
	Estimated Factor	69.6	63.1	48.7	81.2	83.3	78.9	87.1	89.7	89.0
	Plug-in	80.1	83.1	81.4	85.0	89.3	89.6	88.1	91.2	92.4
	WB	87.7	88.1	85.0	90.6	92.2	91.3	92.3	93.3	93.6
	AR-sieve+CSD	89.7	90.5	88.3	92.0	93.0	92.8	92.6	93.9	94.2

In DGP 3, both error terms are heteroskedastic. In DGP 4, the idiosyncratic error term is generated as the autoregressive process of lag 1 for each variable and with heteroskedastic. For coverage rates, the results for estimated factors and plug-ins are based on asymptotic theory. The bootstrap coverage rates use the bootstrap equal-tailed percentile t method.

intervals associated with the factors shift. However, the bias does not have a significant impact on the estimates for the lags of the dependent variable. Additionally, it is worth noting that as we exclude the COVID period, the sign of the estimates associated with the

Table 5: DGP 5 & DGP 6 - Bias and coverage rate of 95% CIs for β

		$N=50$			$N=100$			$N=200$		
		$T=50$	100	200	50	100	200	50	100	200
		$T_H=150$	300	600	150	300	600	150	300	600
	bias									
DGP 5: hetero & CSD	True Factor	-0.00	-0.00	-0.00	-0.00	0.00	-0.00	0.00	-0.00	0.00
	Estimated Factor	-0.09	-0.09	-0.09	-0.06	-0.05	-0.05	-0.03	-0.03	-0.02
	Plug-in	-0.07	-0.06	-0.06	-0.04	-0.04	-0.04	-0.02	-0.02	-0.02
	WB	-0.03	-0.03	-0.03	-0.02	-0.02	-0.02	-0.01	-0.01	-0.01
	AR-sieve+CSD	-0.05	-0.05	-0.05	-0.03	-0.03	-0.03	-0.02	-0.02	-0.02
	95% coverage rate									
	Estimated Factor	80.9	77.1	67.6	87.8	88.1	86.8	90.1	91.9	92.7
	Plug-in	84.6	86.4	86.3	88.5	90.2	91.3	89.8	92.5	93.6
	WB	89.3	87.5	82.7	92.1	91.7	91.0	92.9	93.7	94.2
	AR-sieve+CSD	90.7	90.3	88.8	92.6	92.5	92.7	92.8	93.9	94.7
	bias									
DGP 6: hetero & CSD+AR	True Factor	-0.00	-0.00	-0.00	-0.00	0.00	-0.00	0.00	-0.00	0.00
	Estimated Factor	-0.12	-0.12	-0.12	-0.07	-0.06	-0.06	-0.04	-0.04	-0.03
	Plug-in	-0.06	-0.06	-0.06	-0.04	-0.04	-0.03	-0.02	-0.02	-0.02
	WB	-0.03	-0.03	-0.03	-0.02	-0.02	-0.02	-0.01	-0.01	-0.01
	AR-sieve+CSD	-0.06	-0.06	-0.06	-0.04	-0.04	-0.04	-0.03	-0.02	-0.02
	95% coverage rate									
	Estimated Factor	76.5	70.6	57.1	85.8	85.8	83.0	88.7	90.7	90.9
	Plug-in	82.3	82.3	79.2	86.9	89.4	89.4	89.2	91.8	93.4
	WB	86.3	82.1	73.2	90.7	89.9	88.0	92.3	92.9	93.4
	AR-sieve+CSD	89.6	87.9	84.8	92.1	92.2	91.4	92.8	93.7	94.5

In DGP 5 and 6, both error terms are heteroskedastic. In DGP 5, the idiosyncratic error term contains the cross-sectional dependence. In DGP 6, we impose the dependence in both dimensions for the idiosyncratic error terms. For coverage rates, the results for estimated factors and plug-in are based on asymptotic theory. The bootstrap coverage rates use the bootstrap equal-tailed percentile t method.

two factors is reversed. Previously, the slope coefficient for the aggregated factors was positive, whereas it becomes negative without the COVID period. This suggests that monthly information during the COVID period has a considerable influence on nowcasting the GDP growth rate.

Table 6: Estimation result of long period (1984 Q1 - 2019 Q4)

		$h=2$		$h=1$		$h=0$	
constant		0.87		0.92		0.88	
	Asymptotic	0.70	1.03	0.79	1.06	0.75	1.02
	WB	0.76	1.03	0.84	1.09	0.77	1.02
	AR sieve+CSD	0.79	1.05	0.86	1.11	0.79	1.04
first factor		-1.10		-1.34		-1.27	
	Asymptotic	-1.48	-0.73	-1.67	-1.01	-1.53	-1.00
	WB	-1.52	-0.92	-1.78	-1.20	-1.61	-1.12
	AR sieve+CSD	-1.56	-0.98	-1.83	-1.27	-1.66	-1.16
second factor		0.09		-0.14		-0.01	
	Asymptotic	-0.67	0.84	-0.35	0.07	-0.58	0.56
	WB	-0.13	0.26	-0.40	0.03	-0.23	0.14
	AR sieve+CSD	-0.17	0.24	-0.48	0.02	-0.28	0.13
y_{t-1}		-0.11		-0.19		-0.17	
	Asymptotic	-0.24	0.03	-0.31	-0.06	-0.30	-0.04
	WB	-0.26	0.00	-0.33	-0.10	-0.31	-0.06
	AR sieve+CSD	-0.26	-0.01	-0.35	-0.11	-0.31	-0.06
y_{t-2}		-0.06		-0.09		-0.04	
	Asymptotic	-0.24	0.12	-0.24	0.05	-0.17	0.09
	WB	-0.24	0.08	-0.27	0.03	-0.17	0.08
	AR sieve+CSD	-0.24	0.08	-0.27	0.02	-0.18	0.07
ρ_3		-0.16		-0.14		-0.15	
	Asymptotic	-0.29	-0.02	-0.26	-0.03	-0.26	-0.03
	WB	-0.28	-0.04	-0.26	-0.04	-0.26	-0.04
	AR sieve+CSD	-0.29	-0.04	-0.27	-0.05	-0.26	-0.04

The interval based on the asymptotic theory is obtained by adding and subtracting 1.645 times the heteroskedasticity robust standard errors. The confidence intervals based on bootstrap methods are obtained with equal-tailed bootstrap intervals with a bootstrap number 4999. WB indicates that we use wild bootstrap and AR sieve + CSD indicates that we use the bootstrap algorithm described in Section 3 in the main text.

References

Bai, J. (2003). Inferential theory for factor models of large dimensions. *Econometrica*, *71*(1), 135–171.

Bai, J., & Ng, S. (2006). Confidence intervals for diffusion index forecasts and inference for factor-augmented regressions. *Econometrica*, *74*(4), 1133–1150.

Bai, J., & Ng, S. (2013). Principal components estimation and identification of static factors. *Journal of Econometrics*, *176*(1), 18–29.

Bi, D., Shang, H. L., Yang, Y., & Zhu, H. (2021). *Ar-sieve bootstrap for high-dimensional time series.* Retrieved from `https://arxiv.org/abs/2112.00414`

Gonçalves, S., & Perron, B. (2014). Bootstrapping factor-augmented regression models. *Journal of Econometrics*, *182*(1), 156–173.